Lecture No-11: Topology of Metric Spaces

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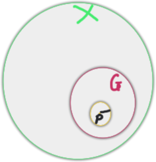
Open sets of Metric Space :-

Let (X, d) be a metric space. Let $G \subseteq X$ is said to be an open set if for every $p \in G$ there exists a positive real number $\varepsilon > 0$ such that

$$B(p, \varepsilon) \subseteq G.$$

Note :- Empty set $\emptyset$ & set X itself is an open sets in X

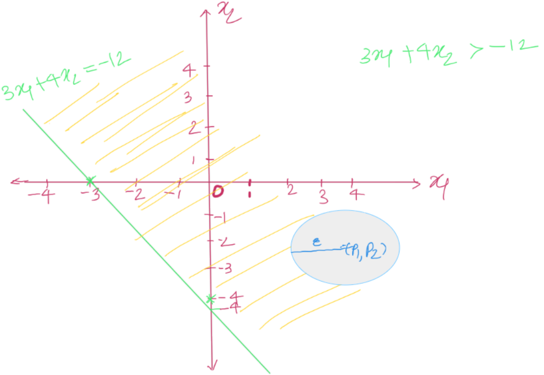


Example 3 :- Let $X = \mathbb{R}^2$ & $d =$ Euclidean metric.
Let $G = \{(x_1, x_2) \in \mathbb{R}^2 \mid 3x + 4y > -12\}$
Then prove that G is open set in X.

Solution :- Geometrically, we can sketch the region representing G. First we sketch the straight line $3x_1 + 4x_2 = -12$

When $x_1 = 0$, $4x_2 = -12 \Rightarrow x_2 = -3$
when $x_2 = 0$, $3x_1 = -12 \Rightarrow x_1 = -4$

We draw a figure


x_2
x_1
$3x_1+4x_2=-12$
$3x_1+4x_2>-12$
ϵ
(p_1,p_2)
4
3
2
1
-4
-3
-2
-1
0
2
3
4
-1
-2
-3
-4

Lecture 11: Metric Spaces

Let $(p_1, p_2) \in G$ be any point.

The perpendicular distance from (p_1, p_2) to the line $3x_1 + 4x_2 = -12$

$$\left| \frac{3p_1 + 4p_2 + 12}{\sqrt{3^2 + 4^2}} \right|$$

Here $a = 3, b = 4$, $c = 12$

$\because$ $\perp^{er}$ distance from pt (a_0, b_0) to line $ax + by + c = 0$ is

$$\left| \frac{a a_0 + b b_0 + c}{\sqrt{a^2 + b^2}} \right|$$

$$= \frac{3p_1 + 4p_2 + 12}{5}$$

Take $\epsilon = \dfrac{3p_1 + 4p_2 + 12}{7} < \dfrac{3p_1 + 4p_2 + 12}{5}$

Note that the denominator of ϵ is obtained by adding the coefficient of numerator $3 + 4 = 7$ & $\dfrac{1}{a+b} < \dfrac{1}{\sqrt{a^2 + b^2}}$

We will prove that $B((p_1, p_2), \epsilon) \subseteq G$

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Let $(y_1, y_2) \in B((p_1, p_2), \epsilon)$

$\Rightarrow d((y_1, y_2),(p_1, p_2)) < \epsilon$ $\quad\because$ d is Euclidean metric

$\Rightarrow \sqrt{(y_1-p_1)^2+(y_2-p_2)^2} < \epsilon$

$\Rightarrow |y_1-p_1| = \sqrt{|y_1-p_1|^2} = \sqrt{|y_1-p_1|^2+|y_2-p_2|^2} < \epsilon$

$\Rightarrow -\epsilon < y_1-p_1 < \epsilon$

$y_1 > p_1-\epsilon$

Similarly, $y_2 > p_2-\epsilon$

$\therefore 3y_1+4y_2 > 3(p_1-\epsilon)+4(p_2-\epsilon) > 3p_1+4p_2-7\epsilon$

$> 3p_1+4p_2-\cancel{7}\times\frac{3p_1+4p_2+12}{\cancel{7}} > 3p_1+4p_2-3p_1-4p_2-12$

$\therefore 3y_1+4y_2 > -12$

$\Rightarrow (y_1, y_2) \in G \Rightarrow B((p_1, p_2), \epsilon) \subseteq G$. Hence G is open.

Example 4: Let $X = \mathbb{R}^2$ & d is Euclidean metric.

Let $G = \{ (x_1, x_2) \in \mathbb{R}^2 \mid -12 < 3x_1 - 2x_2 < 6 \}$.

Then show that G is open in $\mathbb{R}^2$.

Solution :- First we draw a figure.

Consider a st. line $3x_1 - 2x_2 = 6$

$x_1 = 0 \Rightarrow x_2 = -3 \quad \therefore (0, -3)$

$x_2 = 0 \Rightarrow x_1 = 2 \quad \therefore (2, 0)$

Consider a straight line $3x_1 - 2x_2 = -12$

$x_1 = 0 \Rightarrow -2x_2 = -12 \Rightarrow x_2 = 6 \quad (0, 6)$

$x_2 = 0 \Rightarrow 3x_1 = -12 \Rightarrow x_1 = -4 \quad (-4, 0)$

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Let $(p_1, p_2) \in G$ be any arbitrary point.

The perpendicular distance of point (a_0, b_0) to the line $ax + by + c = 0$ is $\left| \frac{aa_0 + bb_0 + c}{\sqrt{a^2+b^2}} \right|$

$\therefore$ The perpendicular distance of point (p_1, p_2) from the line $3x_1 - 2x_2 = -12$ is

$$\left| \frac{3p_1 - 2p_2 + 12}{\sqrt{3^2+2^2}} \right|$$

|||ly, the perpendicular distance of point (p_1, p_2) from the line $3x_1 - 2x_2 = 6$ is

$$\left| \frac{3p_1 - 2p_2 - 6}{\sqrt{3^2+2^2}} \right|$$

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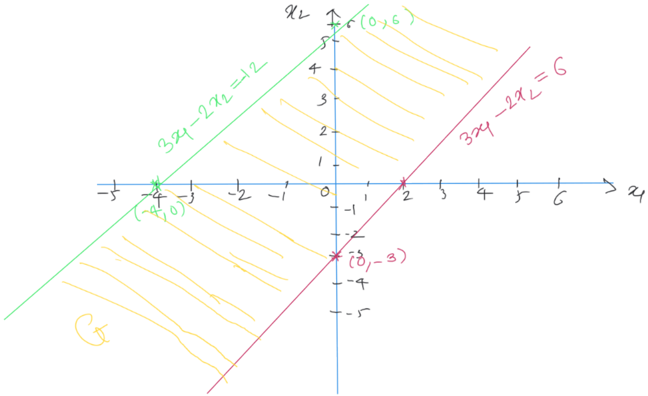

which is respectively $\left|\frac{3p_1-2p_2+12}{\sqrt{13}}\right|$ & $\left|\frac{6-3p_1+2p_2}{\sqrt{13}}\right|$.

We choose ε less than these two values.

Note that $\frac{1}{5} < \frac{1}{\sqrt{13}}$ as $\frac{1}{a+b} < \frac{1}{\sqrt{a^2+b^2}}$

Choose $\varepsilon = \min\left\{\frac{3p_1-2p_2+12}{5}, \frac{6-3p_1+2p_2}{5}\right\}$

We will prove that $B((p_1,p_2),\varepsilon) \subseteq G$

Let $(y_1,y_2) \in B((p_1,p_2),\varepsilon)$

$\Rightarrow d((y_1,y_2),(p_1,p_2)) < \varepsilon$

since d is Euclidean metric

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$$\sqrt{(y_1-p_1)^2+(y_2-p_2)^2} < \varepsilon$$

Consider $|y_1-p_1| = \sqrt{|y_1-p_1|^2} < \sqrt{|y_1-p_1|^2+|y_2-p_2|^2} < \varepsilon$

$\therefore\ |y_1-p_1| < \varepsilon$

$\Rightarrow\ -\varepsilon < y_1-p_1 < \varepsilon$

$\Rightarrow\ p_1-\varepsilon < y_1 < p_1+\varepsilon$

$\therefore\ 3p_1-3\varepsilon < 3y_1 < 3p_1+3\varepsilon$ — (1) (multiplying by 3)

Similarly, $|y_2-p_2| < \varepsilon$

$\Rightarrow\ -\varepsilon < y_2-p_2 < \varepsilon$

$\Rightarrow\ p_2-\varepsilon < y_2 < p_2+\varepsilon$

Multiply by 2 $\quad 2p_2-2\varepsilon < 2y_2 < 2p_2+2\varepsilon$ —— (2)

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Adding ① & ②, we get

$$3p_1 - 3\varepsilon - 2p_2 - 2\varepsilon < 3y_1 - 2y_2 < 3p_1 + 3\varepsilon - 2p_2 + 2\varepsilon$$

$$\therefore \quad 3p_1 - 2p_2 - 5\varepsilon < 3y_1 - 2y_2 < 3p_1 - 2p_2 + 5\varepsilon$$

But $\varepsilon = \dfrac{3p_1 - 2p_2 + 12}{5}$

$$\Rightarrow 3p_1 - 2p_2 - 5 \times \frac{3p_1 - 2p_2 + 12}{5} < 3y_1 - 2y_2 < 3p_1 - 2p_2 + 5 \times \frac{-3p_1 + 2p_2 + 6}{5}$$

$$-12 < 3y_1 - 2y_2 < 6$$

$$\therefore (y_1, y_2) \in G \Rightarrow B((p_1, p_2), \varepsilon) \subseteq G$$

Since (p_1, p_2) is an arbitrary point.

Hence G is an open set in $\mathbb{R}^2$.