My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

Lecture No-8: Topology of Metric Spaces

Santosh Shivlal Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

santosh2maths@gmail.com

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Open Sets of a Metric Spaces

Open Ball in a Metric Space :-

Definition : Open Ball :

Let (X, d) be a metric space. Let $p \in X$ & $\varepsilon > 0$, then the open ball centred at point p & radius ε denoted by $B_d(p, \varepsilon)$ is defined as

$$B_d(p, \varepsilon) = \{ x \in X \mid d(x, p) < \varepsilon \}$$

Some times it is simply denoted as $B(p, \varepsilon)$ in (X, d)

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Examples :-

(1) Let $X = \mathbb{R}$. Let $p \in \mathbb{R}$ & $\varepsilon > 0$ & $d =$ standard metric or absolute value metric.

Solution:-

$$B(p,\varepsilon) = \{x \in \mathbb{R} \mid d(x,p) < \varepsilon\}$$
$$= \{x \in \mathbb{R} \mid |x-p| < \varepsilon\}$$
$$= \{x \in \mathbb{R} \mid -\varepsilon < x-p < \varepsilon\}$$
$$= \{x \in \mathbb{R} \mid p-\varepsilon < x < p+\varepsilon\}$$

$d(x,y) = |x-y|$ standard metric

$\therefore B(p,\varepsilon) = (p-\varepsilon, p+\varepsilon)$

Thus in $(\mathbb{R}, |\ |)$ open balls are open intervals around the point p.

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(2) Let $X = \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$. Then define open ball in $\mathbb{R}^2$ with respective following metric

(i) d_1 = Euclidean metric

(ii) d_2 = sum metric or L^1 metric

(iii) d_3 = Supremum metric

Solution :- Let $p = (p_1, p_2) \in \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ & $\varepsilon > 0$

(i) $B_{d_1}(p, \varepsilon) = B_{d_1}((p_1, p_2), \varepsilon)$

$= \{x \in \mathbb{R}^2 \mid d(x, p) < \varepsilon\}$

$\because x \in \mathbb{R}^2$

$\therefore x = (x_1, x_2)$

$B_{d_1}(p, \varepsilon) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid d((x_1, x_2), (p_1, p_2)) < \varepsilon\}$

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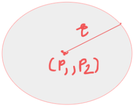
$$\therefore\ B_{d_1}(p,\varepsilon) = \left\{(x_1,x_2)\in \mathbb{R}^2 \;\middle|\; \sqrt{(x_1-p_1)^2+(x_2-p_2)^2} < \varepsilon\right\}$$

$$= \left\{(x_1,x_2)\in \mathbb{R}^2 \;\middle|\; (x_1-p_1)^2+(x_2-p_2)^2 < \varepsilon\right\}$$

= Set of all points inside a circle with center at (p_1,p_2) & radius ε.



$B_{d_1}(p,\varepsilon)$

(ii) Let $X = \mathbb{R}^2$ & d_2 = Sum metric or L^1 metric

$B_{d_2}(p, \varepsilon) = B_{d_2}((p_1, p_2), \varepsilon)$ $\quad \because p \in \mathbb{R}^2 \Rightarrow p = (p_1, p_2)$

$= \{x \in \mathbb{R}^2 \mid d_2(x, p) < \varepsilon\}$

$= \{x \in \mathbb{R}^2 \mid |x_1 - p_1| + |x_2 - p_2| < \varepsilon\}$ $\quad \because x = (x_1, x_2) \in \mathbb{R}^2$

Note that $\underline{|x_1 - p_1|} + \underline{|x_2 - p_2|} < \varepsilon$ gives rise to 4 inequalities

(i) $|x_1 - p_1|$ & $|x_2 - p_2|$ both are positive

$x_1 - p_1 + x_2 - p_2 < \varepsilon$

$\Rightarrow \quad x_1 + x_2 < p_1 + p_2 + \varepsilon$

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(ii) $|x_1 - p_1|$ is negative & $|x_2 - p_2|$ is positive

$\therefore -(x_1 - p_1) + (x_2 - p_2) < \varepsilon$

$\Rightarrow -x_1 + x_2 < p_2 - p_1 + \varepsilon$

(iii) $|x_1 - p_1|$ is positive & $|x_2 - p_2|$ is negative

$\therefore (x_1 - p_1) - (x_2 - p_2) < \varepsilon$

$\Rightarrow x_1 - x_2 < p_1 - p_2 + \varepsilon$

(iv) Both $|x_1 - p_1|$ & $|x_2 - p_2|$ are negative

$\therefore -(x_1 - p_1) - (x_2 - p_2) < \varepsilon$

$\Rightarrow -x_1 - x_2 < \varepsilon - p_1 - p_2$

$\therefore$ $B_{d_2}(p, \varepsilon)$ is a region $\mathbb{R}^2$ bounded by the straight lines

(i) $x_1 + x_2 = p_1 + p_2 + \varepsilon$

(ii) $-x_1 + x_2 = p_2 - p_1 + \varepsilon$

(iii) $x_1 - x_2 = p_1 - p_2 + \varepsilon$

(iv) $-x_1 - x_2 = \varepsilon - p_1 - p_2$

It can be seen that straight lines represented by (i) & (iv) are parallel & straight lines represented by (ii) & (iii) are parallel & hence $B_{d_2}(p, \varepsilon)$ is a parallelogram with (p_1, p_2) at the center (intersecting point of diagonals) with the equation of boundaries of parallelogram represented by equations (i), (ii), (iii) & (iv).

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