My Inspiration
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Lecture No-9: Topology of Metric Spaces

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Lecture 9: Metric Spaces

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Topology of Metric spaces

Unit I : Metric Spaces

Lecture-9

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Open Ball :-

Let (X, d) be a metric space. Let $p \in X$ & $\varepsilon > 0$, then the ball centered at p & radius ε denoted by $B_d(p, \varepsilon)$ or $B(p, \varepsilon)$ is defined as

$$B_d(p, \varepsilon) = \{ x \in X \mid d(x, p) < \varepsilon \}$$

Examples :-

(4) Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ & d = Supremum metric.
Then find $B_d(p, \varepsilon)$

Solution :-

Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ &

d = supremum metric.

$\therefore$ for $x, y \in X = \mathbb{R}^2$, $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

$\because$ $x = (x_1, x_2)$, $y = (y_1, y_2)$

Here we have to find $B(p, \varepsilon)$ w.r.t. supremum metric

$\therefore B(p, \varepsilon) = \{x \in \mathbb{R}^2 \mid d(x, p) < \varepsilon\}$

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$B(p,\varepsilon) = \{ x \in \mathbb{R}^2 \mid d(x,p) < \varepsilon \}$

$= \{ (x_1,x_2) \in \mathbb{R}^2 \mid \max\{ |x_1-p_1|, |x_2-p_2| \} < \varepsilon \}$

$\because X = \mathbb{R}^2 \Rightarrow x = (x_1,x_2)$ & $p = (p_1,p_2)$

Note that there are two possibilities either $|x_1-p_1| < \varepsilon$ or $|x_2-p_2| < \varepsilon$ which gives rise to 4 equations of boundaries of $B(p,\varepsilon)$ which are follows

if $|x_1-p_1| < \varepsilon$ (i) $x_1-p_1 = \varepsilon \Rightarrow x_1 = p_1+\varepsilon$

(ii) $-x_1+p_1 = \varepsilon \Rightarrow x_1 = p_1-\varepsilon$

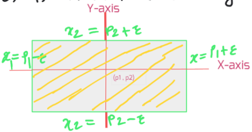
if $|x_2 - p_2| < \varepsilon$ then

(iii) $x_2 - p_2 = \varepsilon \Rightarrow x_2 = p_2 + \varepsilon$

(iv) $-x_2 + p_2 = \varepsilon \Rightarrow x_2 = p_2 - \varepsilon$

We can see that (i) & (ii) represent straight lines parallel to y-axis and equations (iii) & (iv) represents straight lines parallel to x-axis.

$\therefore$ $B(p, \varepsilon)$ is in fact a recatangle.



(5) Let X be any non-empty set & d be the discrete metric where

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \quad \text{for } x, y \in X$$

Then find $B(p,\varepsilon)$ w.r.t. discrete metric.

Solution:- Given X is a non-empty set & d is discrete metric defines as

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

We have to find $B(p,\varepsilon)$

Let $0 < \varepsilon < 1$ & $p \in X$

$$B(p,\varepsilon) = \{x \in X \mid d(p,x) < \varepsilon\}$$
$$= \{p\}$$

as the only values taken by $d(x,y) = 0$ or 1.

If $\varepsilon > 1$, $p \in X$

$$B(p,\varepsilon) = \{x \in X \mid d(x,p) < \varepsilon\}$$
$$= X$$

as every pair of points (x,y) in X are at a distance 1 if $x \neq y$

Thus in a discrete metric space, open balls are either singleton sets or X.

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(6) Let $X = \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$

define $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = \sqrt{9(x_1-y_1)^2 + 16(x_2-y_2)^2}.$$

Find $B((0,0), 1)$.

Solution :– By defination of open ball about point p with radius ε is

$$B(p, \varepsilon) = \{ x \in X \mid d(x, p) < \varepsilon \}$$

Here $p = (0,0)$ & $\varepsilon = 1$

$$\therefore B((0,0), 1) = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid d((x_1, x_2), (0,0)) < 1 \}$$

$$\therefore B((0,0),1) = \{(x_1,x_2)\in\mathbb{R}^2 \mid d((x_1,x_2),(0,0)) < 1\}$$

$$= \{(x_1,x_2)\in\mathbb{R}^2 \mid \sqrt{9(x_1-0)^2+16(x_2-0)^2} < 1\}$$

$$\because d(x,y) = \sqrt{9(x_1-y_1)^2+16(x_2-y_2)^2}$$

$$= \{(x_1,x_2)\in\mathbb{R}^2 \mid \sqrt{9x_1^2+16x_2^2} < 1\}$$

$$= \{(x_1,x_2)\in\mathbb{R}^2 \mid 9x_1^2+16x_2^2 < 1\}$$

$$= \{(x_1,x_2)\in\mathbb{R}^2 \mid \frac{x_1^2}{1/9}+\frac{x_2^2}{1/16} < 1\}$$

$$= \{(x_1,x_2)\in\mathbb{R}^2 \mid \frac{x_1^2}{(\frac{1}{3})^2}+\frac{x_2^2}{(\frac{1}{4})^2} < 1\}$$

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$$B((0,0),1) = \left\{ (x_1,x_2) \in \mathbb{R}^2 \;\middle|\; \underline{\frac{x_1^2}{(\frac{1}{3})^2} + \frac{x_2^2}{(\frac{1}{4})^2} < 1} \right\}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$B((0,0),1)$ = Interior of ellipse $\dfrac{x_1^2}{(\frac{1}{3})^2} + \dfrac{x_2^2}{(1/4)^2} = 1$

with center $(0,0)$ & major axis $= a = \frac{1}{3}$

minor axis $= b = 1/4$.