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Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

Lecture No-12: Topology of Metric Spaces

Santosh Shivlal Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

santosh2maths@gmail.com

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Contents

Sanjeevan Gramin Vidyakiya & Samajik Sahayata Pratishthan's
Arts,Commerce & Science College,Onde
Tal. Vikramgad,Dist. Palghar (MS)-401605
(Affiliated to the University of Mumbai)
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Topology of Metric Spaces

Unit I : Metric Spaces

Lecture - 12

Santosh Shivlal Dhamone
Assistant Professor in Mathematics
Arts Commerce and Science College, Onde

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Theorem :- In any metric space open ball is an open set.

OR

Let (X, d) be a metric space. Every open ball in X is an open set.

Proof :- Let (X, d) be a metric space & $a \in X$.

Let $B(a; s) = G$ is an open ball in X.

Take $x \in B(a, s) = G$

$\therefore d(a, x) < s$ —— (1) by definition

Let $d(a, x) = t$

$\therefore t < s$

$\therefore 0 < s - t$

Choose ε such that $0 < \varepsilon < s - t$ —— $*$

Consider $B(x, \varepsilon)$

If $y \in B(x, \varepsilon)$ then $d(x, y) < \varepsilon$ —— (2)

We have

$d(a, y) \leq d(a, x) + d(x, y)$ $\because$ d is metric & by triangular ineq

By eq^n (1) & (2), we get

$d(a, y) \leq t + \varepsilon$

$\therefore$ $d(a, y) \leq t + s - t$ by $*$

$\therefore$ $d(a, y) < s$

$\therefore$ $y \in B[a, s]$

$\therefore$ $B(x, \varepsilon) \subset B(a, s)$

$\therefore$ $B(a, s) = G$ is open set in X.

Lecture 12: Metric Spaces

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Example :- Show that singlton set in R_d (Discrete metric space) is open set.

Solution :- In metric space R_d, we have

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Let $a \in R_d$. Then

$$B(a,1) = \{x \in R_d \mid d(x,a) < 1\}$$
$$= \{a\}$$

$\therefore B(a,1) = \{a\}$ is open ball in R_d.

By thm, in any metric space open ball is open set

$\therefore \{a\}$ is open set in R_d.

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Note:-

(1) One point set in R^1 is not open set.

Justification :- Let $a \in R^1$. Then $\{a\}$ does not contain any open ball about a.

$\therefore \{a\}$ is not open in R^1.

(2) In any metric space (X, d) both X & $\emptyset$ are open sets.

Theorem :- Arbitrary union of open sets is open

OR

Let (X, d) be a metric space then arbitrary union of open sets is open.

OR

Let $\mathcal{F}$ be a non-empty family of open subsets of X. Then $\bigcup_{G \in \mathcal{F}} G$ is open set in X.

Proof :-

$$\mathcal{F} = \{ G \subset X \mid G \text{ is open set} \}$$

$$\text{Let } H = \bigcup_{G \in \mathcal{F}} G$$

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Then we have to show that H is open set.

If $H = \phi$ then result is true.

If $H \neq \phi$. Then $x \in H$

$$\Rightarrow x \in \bigcup_{G \in \mathcal{F}} G$$

$$\Rightarrow x \in G \text{ for some } G \in \mathcal{F}$$

But G is open set.

$\therefore \exists\ \varepsilon > 0$ such that $B(x, \varepsilon) \subset G$

$$\therefore B(x, \varepsilon) \subset \bigcup_{G \in \mathcal{F}} G = H$$

$\therefore H$ is open set in X.

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Example :- Show that any subset of R_d (Discrete metric space) is open set in R_d.

Solution :- Let $a \in R_d$. Then

$$B(a,1) = \{x \in R_d \mid d(x,a) < 1\}$$
$$= \{a\}$$

$\therefore$ $B(a,1) = \{a\}$ is open set in R_d.

Now let $G \subset R_d$

$$\therefore \quad G = \bigcup_{x \in G} \{x\}$$

$\therefore$ G is expressed as union of open sets

We know that, arbitrary union of open sets is open.

$\therefore$ G is open set in R_d.