My Inspiration
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Lecture No-14: Topology of Metric Spaces

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Lecture 14: Metric Spaces

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Topology of Metric Spaces

Unit I : Metric Spaces

Lecture - 14

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Interior of a Set :-

Definations :-

(1) Interior Point of a Set :-

Let (X, d) be a metric space. Let $A \subseteq X$.
A point $p \in X$ is said to be an interior point of A if there exist $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq A$.

Note that p is an interior point of A, $p \in B(p, \varepsilon) \subseteq A$
$\Rightarrow p \in A$

(2) Interior of a Set :-

The set of all interior points of a set $A \subseteq X$ is called as interior of A & is denoted by

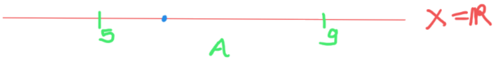
A° or $\text{Int}(A)$.

Note that $A^\circ \subseteq A$

Examples :-

(1) Let $X = \mathbb{R}$, d = Absolute value metric (or standard metric)

Let $A = [5, 9]$. Find the interior of A.



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Solution :- We will check whether all points of A are interior point or not.

Let $5 < p < 9$ be any point.

Let $\varepsilon = \min\{p-5, 9-p\}$. Then

$\varepsilon \leq p-5$ & $\varepsilon \leq 9-p$

We have to prove that $B(p,\varepsilon) \subseteq A = [5,9]$

$$B(p,\varepsilon) = \{x \in \mathbb{R} \mid |p-x| < \varepsilon\}$$
$$= \{x \in \mathbb{R} \mid |x-p| < \varepsilon\}$$
$$= \{x \in \mathbb{R} \mid -\varepsilon < x-p < \varepsilon\}$$

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$$B(p,\varepsilon) = \{x \in \mathbb{R} \mid p-\varepsilon < x < p+\varepsilon\}$$

$$B(p,\varepsilon) = (p-\varepsilon, p+\varepsilon)$$

If $\varepsilon \leq p-5 \Rightarrow p-\varepsilon \geq 5$

& if $\varepsilon \leq 9-p \Rightarrow p+\varepsilon \leq 9$

$$\Rightarrow 5 \leq p-\varepsilon < p+\varepsilon \leq 9$$

$$\Rightarrow (p-\varepsilon, p+\varepsilon) \subseteq [5,9] = A$$

Thus every point p such that $5 < p < 9$ is an interior point.

Now, consider the point 5.

Any open ball centered at 5 is of the form $(5-\varepsilon, 5+\varepsilon)$ wihch contains points < 5.

That is, the portion of the open ball $(5-\varepsilon, 5) \subseteq (5-\varepsilon, 5+\varepsilon)$ contains points outside the interval $[5, 9]$ as

$$(5-\varepsilon, 5) \not\subseteq [5, 9]$$

$\Rightarrow$ 5 is not an interior point.

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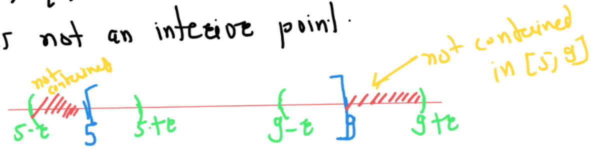
Similarly any open ball centered at 9 is of the form $(9-\varepsilon, 9+\varepsilon)$ and the portion of open ball namely $(9, 9+\varepsilon) \not\subseteq [5, 9]$

$\Rightarrow$ 9 is not an interior point.



$\Rightarrow$ A° = set of all interior points of A

$A^\circ = (5, 9)$

This is required solution.

(2) Let $X = \mathbb{R}$, d = standard metric. Find interiors of

(i) $A = \mathbb{Q}$

(ii) $B = \mathbb{N}$

(iii) $C = \mathbb{Q}^c$

Solution :- (i) Let $p \in \mathbb{Q}$. For any $\varepsilon > 0$,

$B(p, \varepsilon) = (p-\varepsilon, p+\varepsilon)$

contains infinitely many irrational points

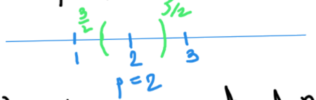
& hence $B(p, \varepsilon) \not\subseteq \mathbb{Q}$ for any $\varepsilon > 0$

$\Rightarrow \mathbb{Q}^\circ = \phi$

(ii) $A = B = \mathbb{N}$.

Let $p \in \mathbb{N}$. For any $\varepsilon > 0$,

$B(p, \varepsilon) = (p - \varepsilon, p + \varepsilon)$ contains non-natural number.



take $\varepsilon = \frac{1}{2} > 0$

$B(p, \varepsilon)$ contains non-natural number like rational & irrational number.

$\therefore$ For any $\varepsilon > 0$,

$$B(p, \varepsilon) \not\subseteq \mathbb{N}$$

$$\Rightarrow \mathbb{N}^{0} = \emptyset$$

(iii) $C = \mathbb{Q}^c$.

Let $p \in \mathbb{Q}^c$. For any $\varepsilon > 0$,

$B(p, \varepsilon) = (p - \varepsilon, p + \varepsilon)$

contains infinitely many rational point

& hence $B(p, \varepsilon) \not\subseteq \mathbb{Q}^c$

$\Rightarrow (\mathbb{Q}^c)^o = \phi$.

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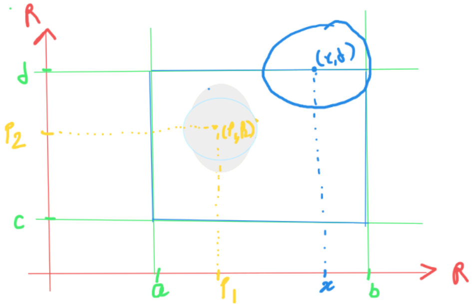
(3) Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, d = standard metric.

Let $S = [a, b] \times [c, d]$.

Find S^0.

Solution :-



Let $(p_1, p_2) \in S$ be such that $a < p_1 < b$ & $c < p_2 < d$.

That is (p_1, p_2) is any point inside rectangle.

Let $\varepsilon = \min\{p_1 - a, b - p_1, p_2 - c, d - p_2\}$

We will prove that $B((p_1, p_2), \varepsilon) \subseteq S$.

Let $(x, y) \in B((p_1, p_2), \varepsilon)$

$$\Rightarrow \sqrt{(x-p_1)^2 + (y-p_2)^2} < \varepsilon$$

$$\Rightarrow |x - p_1| = \sqrt{(x-p_1)^2} \leq \sqrt{(x-p_1)^2 + (y-p_2)^2} < \varepsilon$$

$$\Rightarrow |x - p_1| < \varepsilon$$

$$\Rightarrow -\varepsilon < x - p_1 < \varepsilon$$

$$\Rightarrow p_1 - \varepsilon < x < p_1 + \varepsilon$$

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But $\left.\begin{array}{l} \varepsilon \leq p_1 - a \Rightarrow p_1 - \varepsilon \geq a \\ \varepsilon \leq b - p_1 \Rightarrow p_1 + \varepsilon \leq b \end{array}\right\} (p_1 - \varepsilon, p_1 + \varepsilon) \subseteq (a, b)$

$\Rightarrow a \leq p_1 - \varepsilon < x < p_1 + \varepsilon \leq b$

$\Rightarrow a < x < b$

Similarly, $|y - p_2| < \varepsilon$

$\Rightarrow -\varepsilon < y - p_2 < \varepsilon$

$\Rightarrow p_2 - \varepsilon < y < p_2 + \varepsilon$

$\left.\begin{array}{l} \varepsilon \leq p_2 - c \Rightarrow c \leq p_2 - \varepsilon \\ \varepsilon \leq d - p_2 \Rightarrow \varepsilon + p_2 \leq d_2 \end{array}\right\} (p_2 - \varepsilon, p_2 + \varepsilon) \subseteq (c, d)$

$\Rightarrow c \leq p_2 - \varepsilon < y < p_2 + \varepsilon \leq d \Rightarrow c < y < d$

$\Rightarrow B((P_1, P_2), \varepsilon) \subseteq S$

Thus every point in $\underbrace{(a, b) \times (c, d)}_{\text{rectangle}}$ is an interior point.

Now we will prove that no point on the boundary or border of S is not an interior point.

i.e. (x, d), (x, c) for $a < x < b$
& (a, y), (b, y) for $c < y < d$ } — (1)

are not interior points.

Consider the point $(x, y+\frac{\varepsilon}{2})$

$$d\left((x,y),(x,y+\tfrac{\varepsilon}{2})\right) = \sqrt{(x-x)^2 + (y-(y+\tfrac{\varepsilon}{2}))^2}$$

$$= \sqrt{0^2+(\tfrac{\varepsilon}{2})^2} = \sqrt{(\tfrac{\varepsilon}{2})^2} = \frac{\varepsilon}{2} < \varepsilon$$

$\Rightarrow (x, y+\frac{\varepsilon}{2}) \in B((x,y),\varepsilon)$

But this point has y-coordinate greater than d.

$\Rightarrow (x, d+\frac{\varepsilon}{2}) \notin S \Rightarrow (x, d)$ is not an interior point for $a \leq x \leq b$ as $B((x,d),\varepsilon) \not\subseteq S$ for any $\varepsilon > 0$.

Similarly, we can prove that all points in eqn (1) are not interior points.

$\Rightarrow \operatorname{Int}(S) = S^0 = (a,b) \times (c,d)$