Lecture No-15: Topology of Metric Spaces

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Lecture 15: Metric Spaces

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Topology of Metric Spaces

Unit I : Metric Spaces

Lecture - 15

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My Inspiration
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Properties of Interior of a Set :-

Theorem :- Let (X, d) be a metric Space. Let $A, B \subseteq X$. Then

(1) $\phi^{o} = \phi$

(2) $X^{o} = X$

(3) If $A \subseteq B$ then $A^{o} \subseteq B^{o}$

(4) $(A \cap B)^{o} = A^{o} \cap B^{o}$

(5) $A^{o} \cup B^{o} \subseteq (A \cup B)^{o}$

Proof :-

(1) To prove that $\phi^{\circ} = \phi$.

To show that $\phi^{\circ} = \phi$, enough to show that ϕ contains no point which is not an interior point. This is true clearly as ϕ contains no point.

Hence $\phi^{\circ} = \phi$

(2) To prove that $X^{\circ} = X$.

To show that $X^{\circ} = X$, we will prove that every point of X is an interior point of X.

Let $p \in X$ then for every $\varepsilon > 0$,

$$B(p, \varepsilon) = \{x \in X \mid d(x, p) < \varepsilon\} \subseteq X$$

$\Rightarrow$ p is an interior point of X as every open ball centered at p is a subset of X

$\Rightarrow$ Every point of X is an interior point.

Hence

$$X^{\circ} = X.$$

(3) To Prove that, if $A \subseteq B$ then $A^0 \subseteq B^0$.

Let $A \subseteq B$. Let $a \in A^0 \Rightarrow$ there exist $\varepsilon > 0$ such that

$$B(a, \varepsilon) \subseteq A$$

But $A \subseteq B$

$\Rightarrow \quad B(a, \varepsilon) \subseteq A \subseteq B.$

$\Rightarrow \quad B(a, \varepsilon) \subseteq B$

$\Rightarrow$ a is interior point of B also

$\Rightarrow \quad a \in B^0$

$\Rightarrow \quad A^0 \subseteq B^0$

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(4) To prove that $(A \cap B)^\circ = A^\circ \cap B^\circ$

$A \cap B \subseteq A$ & $A \cap B \subseteq B$

By property (3),

$(A \cap B)^\circ \subseteq A^\circ$ & $(A \cap B)^\circ \subseteq B^\circ$

$\Rightarrow (A \cap B)^\circ \subseteq A^\circ \cap B^\circ$

Now, let $x \in A^\circ \cap B^\circ$

$\Rightarrow x \in A^\circ$ & $x \in B^\circ$

$\Rightarrow$ there exist $\varepsilon_1 > 0$ & $\varepsilon_2 > 0$ such that

$B(x, \varepsilon_1) \subseteq A$ & $B(x, \varepsilon_2) \subseteq B$

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$

We will prove that

$B(x,\epsilon) \subseteq B(x,\epsilon_1)$ & $B(x,\epsilon) \subseteq B(x,\epsilon_2)$

Let $y \in B(x,\epsilon)$

$\Rightarrow d(x,y) < \epsilon \leq \epsilon_1$ & $d(x,y) < \epsilon \leq \epsilon_2$

$\Rightarrow y \in B(x,\epsilon_1)$ & $y \in B(x,\epsilon_2)$

$\Rightarrow B(x,\epsilon) \subseteq B(x,\epsilon_1) \subseteq A$ & $B(x,\epsilon) \subseteq B(x,\epsilon_2) \subseteq B$

$\Rightarrow B(x,\epsilon) \subseteq A \cap B$

$\Rightarrow x \in (A \cap B)^\circ$

$\Rightarrow A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$

Hence $(A \cap B)^\circ = A^\circ \cap B^\circ$

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(5) To Prove that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

$A \subseteq A \cup B$ & $B \subseteq A \cup B$

By property (3)

$A^\circ \subseteq (A \cup B)^\circ$ & $B^\circ \subseteq (A \cup B)^\circ$

$\Rightarrow A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

Remark :-

$A^\circ \cup B^\circ$ need not equal to $(A \cup B)^\circ$.

For example :-

if $X = \mathbb{R}$, d = absolute value metric.

Let $A = [1, 2]$, $B = [2, 3]$ then $A \cup B = [1, 3]$

$A^\circ = (1, 2)$, $B^\circ = (2, 3)$

$\Rightarrow (A^\circ \cup B^\circ) = (1, 2) \cup (2, 3)$

$A^\circ \cup B^\circ = (1, 3) \setminus \{2\}$

$\therefore (A \cup B)^\circ \neq (A^\circ \cup B^\circ$

$[\because (A \cup B)^\circ = (1, 3)]$

Theorem :- Let (X, d) be a metric space. Let $A \subseteq X$, then A is open if and only if

$$A^0 = A$$

Proof :- Suppose $A \subseteq X$ & A is open in X.

Then the largest open set contained in A is itself.

But by Th^m, the interior of any subset A of a metric space (X, d) is the largest open set contained in A.

largest open set contained in A is A^0.

$\Rightarrow A = A^0 = A^0$

Conversely, suppose $A^{o} = A$

Then we have to prove that A is open.

By thm, ...

A^{o} is the largest open set contained in A.

$\Rightarrow A^{o}$ is an open set.

$\Rightarrow A$ is open set as $A = A^{o}$.

Defination :-

Neighbourhood of a point :-

Let (X, d) be a metric space. Let $p \in X$.
A set $U \subseteq X$ is said to be a neighbourhood of the point p, if there exist $\varepsilon > 0$ such that

$$p \in B(p, \varepsilon) \subseteq U.$$

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Theorem :-

If U_1 & U_2 are neighbourhoods of a point $p \in X$, then $U_1 \cap U_2$ is a neighbourhood of p.

Proof :- U_1 is a neighbourhood of $p \in X$

$\Rightarrow$ there exist $\varepsilon_1 > 0$ such that $B(p, \varepsilon) \subseteq U_1$

Similarly, U_2 is a neighbourhood of $p \in X$

$\Rightarrow$ there exist $\varepsilon_2 > 0$ such that $B(p, \varepsilon) \subseteq U_2$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$

We will prove that $B(p, \varepsilon) \subseteq U_1$ & $U_2 = U_1 \cap U_2$.

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Let $x \in B(p, \varepsilon) \Rightarrow d(x, p) < \varepsilon \leq \varepsilon_1$

$\Rightarrow d(x, p) < \varepsilon_1$

$\Rightarrow x \in B(p, \varepsilon_1) \subseteq U_1$

Similarly, $x \in B(p, \varepsilon) \Rightarrow d(x, p) < \varepsilon \leq \varepsilon_2$

$\Rightarrow d(x, p) < \varepsilon_2$

$\Rightarrow x \in B(p, \varepsilon_2) \subseteq U_2$

$\Rightarrow x \in B(p, \varepsilon) \subseteq U_1 \cap U_2$

$\Rightarrow U_1 \cap U_2$ is a neighbourhood of p.

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Corollary :-

If $U_1, U_2, U_3, \ldots, U_n$ are n neighbourhoods of $p \in X$ then $U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_n$ is also a neighbourhood of p.

Example :-

(1) Every open set is a neighbourhood of every point in it.

(2) $1 \in [0, 2]$ and $[0, 2]$ is a neighbourhood of 1, as $1 \in \left(\frac{1}{2}, \frac{3}{2}\right) \subseteq [0, 2]$ i.e. $[0, 2]$ contains an open ball centered at 1.