Lecture No-16: Topology of Metric Spaces

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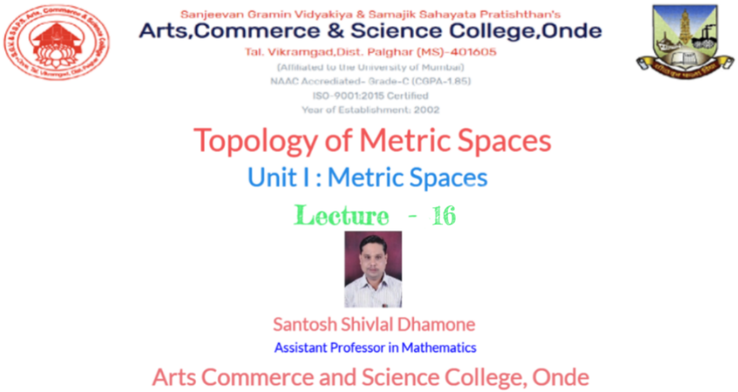

My Inspiration Shri. V.G. Patil Saheb
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Diameter of a Set and Hausdorff Property:-

Defination of Diameter of a Set :-

Let (X, d) be a metric space. Let A be a non-empty subset of X. The diameter of a set A is denoted by $\delta(A)$ is defined as

$$\delta(A) = \begin{cases} \sup\{d(x,y) \mid x, y \in A\} & \text{if supremum of the set exist} \\ \infty & \text{if supremum of the set does not exist.} \end{cases}$$

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Examples :-

(1) If (X, d) is a metric space and let $x_0 \in X$ be a fixed point.

Let $A = \{x_0\} \subseteq X$ then

$$\delta(A) = \sup\{d(x_0, x_0) \mid x_0 \in A\} = 0$$

$$\therefore \delta(A) = 0$$

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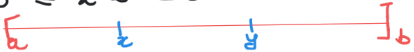
(2) Let $X = \mathbb{R}$, $d(x, y) = |x - y|$ be the absolute value metric.
Let $A = [a, b]$ be a bounded interval,
then find $\delta(A)$.

Sol^n :- $\delta(A) = \sup\{ d(x, y) \mid x, y \in A \}$

$$= \sup\{ |x - y| \mid x, y \in [a, b] \}$$

Note that $a \leq x \leq b \quad \forall\ x \in [a, b]$

$$\Rightarrow \quad 0 \leq x - a \leq b - a$$



It can be seen that $0 \le |x-y| \le b-a \quad \forall x \in [a,b]$

$\Rightarrow \sup\{|x-y| \mid x, y \in [a,b]\} = b-a$

$\Rightarrow \quad \delta(A) = b-a$

(3) Let $X = \mathbb{R}^2$ and d be the Euclidean metric.

Let $A = \{(x_1, x_2) \mid x_1^2 + x_2^2 \le 1\}$

= set of all points in the interior of a circle with center at origin & radius 1.

Find $\delta(A)$.

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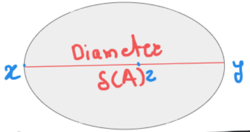
Solution :- Let $x=(x_1,x_2)$, $y=(y_1,y_2)$ in $\mathbb{R}^2$.

The longest chord in a circle is its diameter.



$\because$ d is Euclidean metric

$\Rightarrow \quad 0 \le d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

By triangular inequality

$d(x,y) \le d(x,z)+d(z,y)$ where $z=(0,0)$

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$$\therefore d(x, y) \leq \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$$

$$\leq 1 + 1 \quad \forall\ x, y \in A$$

$$\leq 2 \qquad \because x_1^2 + x_2^2 \leq 1 \ \& \ y_1^2 + y_2^2 \leq 1.$$

$$\Rightarrow \delta(A) = \sup\{d(x, y) \mid x, y \in A\}$$

$$\boxed{\delta(A) = 2}$$

Remark :-

$$\delta(B(p, \varepsilon)) = 2\varepsilon = \varepsilon + \varepsilon$$

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Hausdorff Property of Metric Space :-

Theorem :- Let (X, d) be a metric space. Let $x, y \in X$, $x \neq y$, then there exists open sets U, V in X such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Proof :- Given $x \neq y$

$\Rightarrow d(x, y) > 0$ by defn of metric space

Let $\varepsilon = d(x, y)$.

Clearly $\varepsilon > 0$

Let $U = B(x, \frac{\varepsilon}{3})$ and $V = B(y, \frac{\varepsilon}{3})$

U and V are open balls in metric space X.

$\therefore$ By thm, U and V are open sets in X.

$d(x, x) = 0 < \frac{\varepsilon}{3}$

$\Rightarrow x \in B(x, \frac{\varepsilon}{3}) = U$

Similarly, $d(y, y) = 0 < \frac{\varepsilon}{3}$

$\Rightarrow y \in B(y, \frac{\varepsilon}{3}) = V$

We will show that $U \cap V = \phi$

But $d(x, y) = \varepsilon$

$\Rightarrow \quad \varepsilon < \frac{2\varepsilon}{3}$

$\Rightarrow \quad 3\varepsilon < 2\varepsilon$

$\Rightarrow \quad 3 < 2$ ——— Impossible

which is a contradiction.

Hence our assumption $U \cap V \neq \phi$ is wrong.

& hence $U \cap V = \phi$ which completes the proof.

Suppose if possible let $U \cap V \neq \phi$

let $z \in U \cap V$

$\Rightarrow z \in U$ & $z \in V$

$z \in U = B(x, \frac{\varepsilon}{3}) \Rightarrow d(x, z) < \frac{\varepsilon}{3}$

Illy, $z \in V = B(y, \frac{\varepsilon}{3}) \Rightarrow d(y, z) < \frac{\varepsilon}{3}$

By triangular inequality of metric spaces

$$d(x, y) \leq d(x, z) + d(z, y)$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$$

But $d(x, y) = \varepsilon$