



Art's Commerce and Science College, Onda

Tal:- Vikramgad, Dist:- Palghar

Topology of Metric Spaces

My Inspiration
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Lecture No-16: Topology of Metric Spaces

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Topology of Metric Spaces

Unit I : Metric Spaces

Lecture - 16



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Diameter of a Set and Hausdorff Property:-

Definition of Diameter of a Set :-

Let (X, d) be a metric space. Let A be a non-empty subset of X . The diameter of a set A is denoted by $\delta(A)$ is defined as

$$\delta(A) = \begin{cases} \sup \{ d(x, y) \mid x, y \in A \} & \text{if supremum of the set exist} \\ \infty & \text{if supremum of the set does not exist.} \end{cases}$$



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Examples :-

(1) If (X, d) is a metric space and let $x_0 \in X$ be a fixed point.

Let $A = \{x_0\} \subseteq X$ then

$$\delta(A) = \sup \{d(x_0, x_0) \mid x_0 \in A\} = 0$$

$$\therefore \delta(A) = 0$$



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(2) Let $X = \mathbb{R}$, $d(x, y) = |x - y|$ be the absolute value metric.

Let $A = [a, b]$ be a bounded interval, then find $\delta(A)$.

$$\begin{aligned}\text{Sol}^n:- \delta(A) &= \sup \{ d(x, y) \mid x, y \in A \} \\ &= \sup \{ |x - y| \mid x, y \in [a, b] \}\end{aligned}$$

Note that $a \leq x \leq b \quad \forall x \in [a, b]$

$$\Rightarrow 0 \leq x - a \leq b - a$$





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It can be seen that $0 \leq |x-y| \leq b-a \quad \forall x, y \in [a, b]$
 $\Rightarrow \sup \{ |x-y| \mid x, y \in [a, b] \} = b-a$
 $\Rightarrow \delta(A) = b-a$

(3) Let $X = \mathbb{R}^2$ and d be the Euclidean metric.
Let $A = \{ (x_1, x_2) \mid x_1^2 + x_2^2 \leq 1 \}$
= set of all points in the interior of
a circle with center at origin of
radius 1.
Find $\delta(A)$.

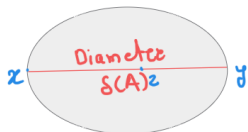


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Solution :- Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ in \mathbb{R}^2 .
The longest chord in a circle is its diameter.



$\therefore d$ is
Euclidean metric

$$\Rightarrow 0 \leq d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

By triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y) \quad \text{where } z = (0, 0)$$



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$$\therefore d(x, y) \leq \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$$

$$\leq 1 + 1 \quad \forall x, y \in A$$

$$\leq 2$$

$$\therefore x_1^2 + x_2^2 \leq 1 \quad \&$$

$$y_1^2 + y_2^2 \leq 1.$$

$$\Rightarrow \delta(A) = \sup \{d(x, y) \mid x, y \in A\}$$

$$\boxed{\delta(A) = 2}$$

Remark \therefore

$$\delta(B(p, \varepsilon)) = 2\varepsilon = \varepsilon + \varepsilon$$



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Hausdorff Property of Metric Space :-

Theorem :- Let (X, d) be a metric space. Let $x, y \in X$,
 $x \neq y$, then there exists open sets
 U, V in X such that $x \in U$, $y \in V$ and
 $U \cap V = \emptyset$.

Proof :- Given $x \neq y$
 $\Rightarrow d(x, y) > 0$
Let $\varepsilon = d(x, y)$.
Clearly $\varepsilon > 0$

by defⁿ of metric space



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$$\text{Let } U = B(x, \frac{\epsilon}{3}) \text{ and } V = B(y, \frac{\epsilon}{3})$$

U and V are open balls in metric space X .

\therefore By thⁿ, U and V are open sets in X .

$$d(x, x) = 0 < \frac{\epsilon}{3}$$

$$\Rightarrow x \in B(x, \frac{\epsilon}{3}) = U$$

Similarly, $d(y, y) = 0 < \frac{\epsilon}{3}$

$$\Rightarrow y \in B(y, \frac{\epsilon}{3}) = V$$

We will show that $U \cap V = \emptyset$



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$$\text{But } d(x, y) = \varepsilon$$

$$\Rightarrow \varepsilon < \frac{2\varepsilon}{3}$$

$$\Rightarrow 3\varepsilon < 2\varepsilon$$

$$\Rightarrow 3 < 2 \text{ ——— Impossible}$$

which is a contradiction.

Hence our assumption $U \cap V \neq \emptyset$ is wrong.
& hence $U \cap V = \emptyset$ which completes the proof.



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Suppose if possible let $U \cap V \neq \emptyset$

let $z \in U \cap V$

$\Rightarrow z \in U$ & $z \in V$

$z \in U = B(x, \frac{\epsilon}{3}) \Rightarrow d(x, z) < \frac{\epsilon}{3}$

Similarly, $z \in V = B(y, \frac{\epsilon}{3}) \Rightarrow d(y, z) < \frac{\epsilon}{3}$

By triangle inequality of metric spaces

$$\begin{aligned}d(x, y) &\leq d(x, z) + d(z, y) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}\end{aligned}$$

But $d(x, y) = \epsilon$