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Lecture No-10: System of Linear Equations and Matrices

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Contents

Homogeneous systems
Properties of the homogeneous system for Amn
Examples

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Linear Algebra - I

Unit I : System of Linear Equations and Matrices

Lecture - 10

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5 Homogeneous systems

Definition A *homogeneous* (ho-mo-geen'-ius) system of linear algebraic equations is one in which all the numbers on the right hand side are equal to 0:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ \vdots \quad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

In matrix form, this reads $A\mathbf{x} = \mathbf{0}$, where A is $m \times n$,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1},$$

and $\mathbf{0}$ is $n \times 1$. The homogenous system $A\mathbf{x} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$. It follows that any homogeneous system of equations is alwasy consistent. Any non-zero solutions, if they exist, are said to be *non-trivial* solutions. These may or may not exist. We can find out by row reducing the corresponding augmented matrix $(A|\mathbf{0})$.

Example: Given the augmented matrix

$$(A|\mathbf{0}) = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ -2 & -3 & 4 & 5 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{pmatrix},$$

row reduction leads quickly to the echelon form

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that nothing happened to the last column — row operations don't do anything to a column of zeros. In particular, doing a row operation on a system of homogeneous equations doesn't change the fact that it's homogeneous. For this reason, when working with homogeneous systems, we'll just use the matrix A. The echelon form of A is

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$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here, the leading variables are x_1 and x_2, while x_3 and x_4 are free variables, since there are no leading entries in the third or fourth columns. Continuing along, we obtain the Gauss-Jordan form (You are working out all the details on your scratch paper as we go along, aren't you!?)

$$\begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

No further simplification is possible; any further row operations will destroy the Gauss-Jordan structure of the columns with leading entries. The resulting system of equations reads

$$\begin{aligned} x_1 - 8x_3 - 7x_4 &= 0 \\ x_2 + 4x_3 + 3x_4 &= 0, \end{aligned}$$

In principle, we're done in the sense that we have the solution in hand. However, it's customary to rewrite the solution in vector form so that its properties are more clearly displayed. First, we solve for the leading variables; everything else goes on the right hand side of the equations:

$$\begin{aligned} x_1 &= 8x_3 + 7x_4 \\ x_2 &= -4x_3 - 3x_4. \end{aligned}$$

Assigning any values we choose to the two free variables x_3 and x_4 gives us a solution to the original homogeneous system. This is, of course, whe the variables are called "free". We can distinguish the free variables from the leading variables by denoting them as s, t, u, etc.

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Santosh Shivlal Dhamone

Thus, setting $x_3 = s$, $x_4 = t$, we rewrite the solution in the form

$$\begin{aligned} x_1 &= 8s + 7t \\ x_2 &= -4s - 3t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

Better yet, the solution can also be written in matrix (vector) form as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \qquad (1)$$

We call (1) the *general solution* to the homogeneous equation. The notation is misleading, since the left hand side $\mathbf{x}$ looks like a single vector, while the right hand side clearly represents an infinite collection of objects with 2 degrees of freedom. We'll address this later in the lecture.

We won't do it here, but If we were to carry out the above procedure on a general homogeneous system $A_{m\times n}\mathbf{x} = \mathbf{0}$, we'd establish the following facts:

5.1 Properties of the homogenous system for A_{mn}

- The number of leading variables is $\leq \min(m, n)$.
- The number of non-zero equations in the echelon form of the system is equal to the number of leading entries.
- The number of free variables plus the number of leading variables $= n$, the number of columns of A.
- The homogenous system $A\mathbf{x} = \mathbf{0}$ has *non-trivial* solutions if and only if there are free variables.

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- If there are more unknowns than equations, the homogeneous system *always* has non-trivial solutions. Why? This is one of the few cases in which we can tell something about the solutions without doing any work.
- A homogeneous system of equations is always consistent (i.e., always has at least one solution).

  Exercise: What sort of geometric object does $\mathbf{x}_H$ represent?

There are two other fundamental properties:

1. Theorem: If $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{0}$, then so is $c\mathbf{x}$ for any real number c.

   Proof: $\mathbf{x}$ is a solution means $A\mathbf{x} = \mathbf{0}$. But $Ac\mathbf{x} = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}$, so $c\mathbf{x}$ is also a solution.

2. Theorem: If $\mathbf{x}$ and $\mathbf{y}$ are two solutions to the homogeneous equation, then so is $\mathbf{x} + \mathbf{y}$.

   Proof: $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

These two properties constitute the famous *principle of superposition* which holds for homogeneous systems (but NOT for inhomogeneous ones).

Definition: if $\mathbf{x}$ and $\mathbf{y}$ are two vectors and s and t two scalars, then $s\mathbf{x} + t\mathbf{y}$ is called a *linear combination* of $\mathbf{x}$ and $\mathbf{y}$.

Example: $3\mathbf{x} - 4\pi\mathbf{y}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$.

We can restate the superposition principle as:

Superposition principle: if $\mathbf{x}$ and $\mathbf{y}$ are two solutions to the homogenous equation $A\mathbf{x} = \mathbf{0}$, then any linear combination of $\mathbf{x}$ and $\mathbf{y}$ is also a solution.

Remark: This is just a compact way of restating the two properties: If $\mathbf{x}$ and $\mathbf{y}$ are solutions, then by property 1, $s\mathbf{x}$ and $t\mathbf{y}$ are also solutions. And by property 2, their sum $s\mathbf{x} + t\mathbf{y}$ is a solution. Conversely, if $s\mathbf{x} + t\mathbf{y}$ is a solution to the homogeneous equation for all s, t, then taking $t = 0$ gives property 1, and taking $s = t = 1$ gives property 2.

You have seen this principle at work in your calculus courses.

Example: Suppose $\phi(x, y)$ satisfies LaPlace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

We write this as

$$\Delta \phi = 0, \quad \text{where } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The *differential operator* Δ has the same property as matrix multiplication, namely: if $\phi(x, y)$ and $\psi(x, y)$ are two differentiable functions, and s and t any two real numbers, then

$$\Delta(s\phi + t\psi) = s\Delta\phi + t\Delta\psi.$$

It follows that if ϕ and ψ are two solutions to Laplace's equation, then any linear combination of ϕ and ψ is also a solution. The principle of superposition also holds for solutions to the wave equation, Maxwell's equations in free space, and Schrödinger's equation in quantum mechanics.

Example: Start with "white" light (e.g., sunlight); it's a collection of electromagnetic waves which satisfy Maxwell's equations. Pass the light through a prism, obtaining red, orange, . . . , violet light; these are also solutions to Maxwell's equations. The original solution (white light) is seen to be a superposition of many other solution, corresponding to the various different colors. The process can be reversed to obtain white light again by passing the different colors of the spectrum through an inverted prism.

Referring back to the example (see Eqn (1)), if we set

$$\mathbf{x} = \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{y} = \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix},$$

then the susperposition principle tells us that any linear combination of $\mathbf{x}$ and $\mathbf{y}$ is also a solution. In fact, these are *all* of the solutions to this system.

Definition: We write

$$\mathbf{x}_H = \{s\mathbf{x} + t\mathbf{y} : \forall \text{ real } s, t\}$$

and say that $\mathbf{x}_H$ is the *general solution* to the homogeneous system $A\mathbf{x} = \mathbf{0}$.