My Inspiration
Shri. V.G. Patil
Saheb

Lecture No-11: System of Linear Equations and Matrices

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Linear Algebra - I

Unit I : System of Linear Equations and Matrices

Lecture - 11

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Lecture 11: System of Linear Equations and Matrices

6 The Inhomogeneous system $A\mathbf{x} = \mathbf{y}$, $\mathbf{y} \neq 0$

Definition: The system $A\mathbf{x} = \mathbf{y}$ is *inhomogeneous* if it's not homogeneous.

Mathematicians love definitions like this! It means of course that the vector $\mathbf{y}$ is not the zero vector. And this means that at least one of the equations has a non-zero right hand side.

As an example, we can use the same system as in the previous lecture, except we'll change the right hand side to something non-zero:

$$\begin{aligned} x_1 + 2x_2 - x_4 &= 1 \\ -2x_1 - 3x_2 + 4x_3 + 5x_4 &= 2 \\ 2x_1 + 4x_2 - 2x_4 &= 3 \end{aligned}.$$

Those of you with sharp eyes should be able to tell at a glance that this system is *inconsistent* — that is, there are *no* solutions. Why? We're going to proceed anyway because this is hardly an exceptional situation.

The augmented matrix is

$$(A\vdots\mathbf{y}) = \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 3 \end{pmatrix}.$$

We can't discard the 5th column here since it's not zero. The row echelon form of the augmented matrix is

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 1 & 4 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

And the reduced echelon form is

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The third equation, from either of these, now reads

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1, \text{ or } 0 = 1.$$

This is false! How can we wind up with a false statement? The actual reasoning that led us here is this: *If* the original system has a solution, then performing elementary row operations gives us an equivalent system of equations which has the same solution. But this equivalent system of equations is *inconsistent*. It has no solutions; that is *no* choice of $x_1, \ldots, x_4$ satisfies the equation. So the original system is also inconsistent.

In general: *If the echelon form of $(A\vdots\mathbf{y})$ has a leading 1 in any position of the last column, the system of equations is inconsistent.*

Now it's not true that any inhomogenous system with the same matrix A is inconsistent. It depends completely on the particular $\mathbf{y}$ which sits on the right hand side. For instance, if

$$\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},$$

then (work this out!) the echelon form of $(A\vdots\mathbf{y})$ is

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 1 & 4 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Since this is consistent, we have, as in the homogeneous case, the leading variables x_1 and x_2, and the free variables x_3 and x_4. Renaming the free variables by s and t, and writing out the equations solved for the leading variables gives us

$$\begin{aligned} x_1 &= 8s + 7t - 7 \\ x_2 &= -4s - 3t + 4 \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

This looks like the solution to the homogeneous equation found in the previous section except for the additional scalars -7 and $+4$ in the first two equations. If we rewrite this using vector notation, we get

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -7 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

Compare this with the general solution $\mathbf{x}_H$ to the homogenous equation found before. Once again, we have a 2-parameter family of solutions. We can get what is called a *particular* solution by making some specific choice for s and t. For example, taking $s = t = 0$, we get the particular solution

$$\mathbf{x}_p = \begin{pmatrix} -7 \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

We can get other particular solutions by making other choices. Observe that the *general solution to the inhomogeneous system* worked out here can be written in the form $\mathbf{x} = \mathbf{x}_H + \mathbf{x}_p$. In fact, this is true in general:

Theorem: Let $\mathbf{x}_p$ and $\mathbf{y}_p$ be two solutions to $A\mathbf{x} = \mathbf{y}$. Then their difference $\mathbf{x}_p - \mathbf{y}_p$ is a

Theorem :- If xp & yp are solutions of non-homogeneous system of linear equations $Ax = y$ then $xp - yp$ is also

solution to the homogeneous equation $Ax = 0$. The *general solution* to $Ax = y$ can be written as $\mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_h$ denotes the general solution to the homogeneous system.

$Ax = y$

$\because$ x_p is soln

$\Rightarrow Ax_p = y$

& if y_p is soln

$\Rightarrow Ay_p = y$

Proof: Since $\mathbf{x}_p$ and $\mathbf{y}_p$ are solutions, we have $A(\mathbf{x}_p - \mathbf{y}_p) = A\mathbf{x}_p - A\mathbf{y}_p = \mathbf{y} - \mathbf{y} = 0$. So their difference solves the homogeneous equation. Conversely, given a particular solution $\mathbf{x}_p$, then the entire set $\mathbf{x}_p + \mathbf{x}_h$ consists of solutions to $A\mathbf{x} = \mathbf{y}$: if $\mathbf{z}$ belongs to $\mathbf{x}_h$, then $A(\mathbf{x}_p + \mathbf{z}) = A\mathbf{x}_p + A\mathbf{z} = \mathbf{y} + 0 = \mathbf{y}$ and so $\mathbf{x}_p - \mathbf{z}$ is a solution to $A\mathbf{x} = \mathbf{y}$.

Going back to the example, suppose we write the general solution to $A\mathbf{x} = \mathbf{y}$ in the vector form

$$\mathbf{x} = s\mathbf{v}_1 - t\mathbf{v}_2 + \mathbf{x}_p,$$

where

$$\mathbf{v}_1 = \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \mathbf{x}_p = \begin{pmatrix} -7 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

Now we e can get another particular solution to the system by taking $s = 1$, $t = 1$. This gives

$$y_p = \begin{pmatrix} 8 \\ -3 \\ 1 \\ 1 \end{pmatrix}.$$

We can rewrite the general solution as

$$\begin{aligned} \mathbf{x} &= (s - 1 + 1)\mathbf{v}_1 + (t - 1 + 1)\mathbf{v}_2 + \mathbf{x}_p \\ &= (s - 1)\mathbf{v}_1 + (t - 1)\mathbf{v}_2 + \mathbf{y}_p \\ &= \hat{s}\mathbf{v}_1 + \hat{t}\mathbf{v}_2 + \mathbf{y}_p \end{aligned}$$

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As $\hat{s}$ and $\hat{t}$ run over all possible pairs of real numbers we get exactly the same set of solutions as before. So the general solution can be written as $\mathbf{y}_p + \mathbf{x}_h$ as well as $\mathbf{x}_p + \mathbf{x}_h$! This is a bit confusing until you remember that these are *sets* of solutions, rather than single solutions; $(\hat{s}, \hat{t})$ and (s, t) are just different sets of coordinates. But running through either set of coordinates (or parameters) produces the same set.

Remarks

- Those of you taking a course in differential equations will encounter a similar situation: the general solution to a linear differential equation has the form $y = y_p + y_h$, where y_p is any particular solution to the DE, and y_h denotes the set of all solutions to the homogeneous DE.

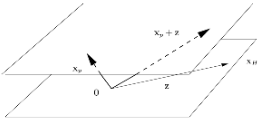

Figure 1: The lower plane (the one passing through 0) represents $\mathbf{x}_H$. Given the particular solution $\mathbf{x}_p$ and a $\mathbf{z}$ in $\mathbf{x}_H$, we get another solution to the inhomogeneous equation. As $\mathbf{z}$ varies in $\mathbf{x}_H$, we get all the solutions to $A\mathbf{x} = \mathbf{y}$.

- We can visualize the general solutions to the homogeneous and inhomogeneous equations we've worked out in detail as follows. The set $\mathbf{x}_H$ is a 2-plane in $\mathbb{R}^4$ which goes through the origin since $\mathbf{x} = \mathbf{0}$ is a solution. The general solution to $A\mathbf{x} = \mathbf{y}$ is obtained by adding the vector $\mathbf{x}_p$ to every point in this 2-plane. Geometrically, this gives another 2-plane parallel to the first, but *not* containing the origin (since $\mathbf{x} = \mathbf{0}$ is not a solution to $A\mathbf{x} = \mathbf{y}$ unless $\mathbf{y} = \mathbf{0}$). Now pick *any* point in this parallel 2-plane and add to it all the vectors in the 2-plane corresponding to $\mathbf{x}_h$. What do you get? You get the same parallel 2-plane! This is why $\mathbf{x}_p + \mathbf{x}_h = \mathbf{y}_p + \mathbf{x}_h$.