Lecture No-2: Metric Spaces

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Defination

Let X be an arbitrary set which could consist of vectors in R^n, functions, sequences, matrices, etc. We want to endow this set with a metric; i.e a way to measure distance between elements of X. A distance or metric is a function $d : X \times X \to R$ such that if we take two elements $x_1, x_2 \in X$ the number $d(x_1, x_2)$ gives us the distance between them.

However, not just any function may be considered a metric: as we will see in the formal definition, a distance needs to satisfy certain properties.

Definition of Metric

First we discuss *Definition of Metric*

Metric:-

Let X be a non-empty set and R be a set of real numbers.

Let $d : X \times X \to R$ be a function, then "d" is called "metric" on$X$, if "d" satisfies each of the following four conditions:

1. $d(x_1, x_2) \geq 0 \quad \forall x_1, x_2 \in X$
2. $d(x_1, x_2) = 0 \iff x_1 = x_2 \quad \forall x_1, x_2 \in X$
3. Symmetric Property: $d(x_1, x_2) = d(x_2, x_1) \quad \forall x_1, x_2 \in X$
4. Triangular Inequality: $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3) \quad \forall x_1, x_2, x_3 \in X$

Notes and Some Standard Metric

Notes

Note:- The non-negative real number $d(x_1, x_2)$ is called distance between points $x_1 and x_2$ in the metric "d"

Usual Metric on R:-

Let $d : R \times R \to R$ be a metric on R given by $d(x_1, x_2) = |x_1 - x_2|$. Then "$d$" is called a usual metric on R and (R, d) is called usual metric space.

Usual Metric on R^2:-

Let $d : R^2 \times R^2 \to R$ be a metric on R^2 given by $d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Then "$d$" is called a usual metric on R^2 and (R^2, d) is called usual metric space.

Usual Metric on R^3:-

Let $d : R^3 \times R^3 \to R$ be a metric on R^3 given by $d[(x_1, y_1, z_1), (x_2, y_2, z_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.
Then "d" is called a usual metric on R^3 and (R^3, d) is called usual metric space.

Examples

Example 1: Let the function d be defined as $d : R \times R \to [0, \infty)$ for $x, y \in R$ such that $d(x, y) = |x - y|$. Then show that d is metric for the set R.

Solution: Here function $d : R \times R \to [0, \infty)$ is defined as $d(x, y) = |x - y|$; for $x, y \in R$

(1) For $x, y \in R$. Let x $\neq y$

$\therefore$ x-y $\neq 0$

$$|x - y| > 0$$

$$d(x, y) > 0 \quad \forall x, y \in R$$

(2) For $x \in R$. we have

$$|x - x| = 0$$

$$\therefore \quad d(x,x) = 0$$

(3) Let $d(x, y) = |x - y|$

$$= |-(y - x)| = |y - x|$$

$d(x, y) = d(y, x) ... \forall x, y \in R$

(4) Let $x, y, z \in R$. Now

$d(x, y) = |x - y|$

$d(x, y) = |x - z + z - y|$

$d(x, y) \leq |x - z| + |z - y|$

$d(x, y) \leq d(x, z) + d(z, y)$

By (1),(2),(3) and (4), we get

d is metric for set R.

Examples

Example 2: Define $d : R \times R \to [0, \infty)$ as
$d(x, y) = 0$ if $x = y$
$d(x, y) = 1$ if $x \neq y$.
Then show that d is metric for the set R.
Solution: Here function $d : R \times R \to [0, \infty)$; for $x, y \in R$ is defined as
$d(x, y) = 0$ if $x = y$
$d(x, y) = 1$ if $x \neq y$.

(1) For $x, y \in R$. Let $x \neq y$; then by definition of function, we have
$\therefore$ d(x,y)=1> 0
$$d(x, y) > 0 \quad \forall x, y \in R$$

(2) For $x \in R$.By definition we have

$\therefore$ d(x,x)= 0

(3) For $x, y \in R$. Let $d(x, y) = 1$

$\implies$ d(y,x)=1

$d(x, y) = d(y, x)...\forall x, y \in R$

(4) For $x, y, z \in R$. Let $x = y = z$

$d(x, y) = 0, d(x, z) = 0, d(z, y) = 0$

$d(x, y) = d(x, z) + d(z, y)$....(i)

Let $x \neq y \neq z$.Then

$d(x, y) = 1, d(x, z) = 1, d(z, y) = 1$

$d(x, y) < d(x, z) + d(z, y)$....(ii)

Let $x = y \quad y \neq z$.Then
$d(x, y) = 0, d(x, z) = 1, d(z, y) = 1$
$d(x, y) < d(x, z) + d(z, y)$....(iii)
By (i),(ii) (iii) we have
$d(x, y) \leq d(x, z) + d(z, y)$....(4)
By (1),(2),(3) and (4), we get
d is metric for set R.
Remark:(i) The metric
$d(x, y) = 0$ if $x = y$
$d(x, y) = 1$ if $x \neq y$.
on the set R is called discrete metric. It is denoted by "d".
(ii)The metric space $(R, d) = R_d$ is called discrete metric space.

Examples

Example 3: Define $d : R^2 \times R^2 \to [0, \infty)$ as
$d(x, y) = |x_1 - x_2| + |y_1 - y_2|$
where $x = (x_1, y_1)$ and $y = (x_2, y_2)$ are in R^2.
Then show that d is metric for R^2.

Solution: Here function $d : R^2 \times R^2 \to [0, \infty)$ as
$d(x, y) = |x_1 - x_2| + |y_1 - y_2|$
where $x = (x_1, y_1)$ and $y = (x_2, y_2)$ are in R^2.

(1) For $x, y \in R^2$. Let $x \neq y$; then $(x_1, y_1) \neq (x_2, y_2)$
$\therefore$ *either* $x_1 \neq x_2$ or $y_1 \neq y_2$ or both
either $|x_1 - x_2| > 0$ or $|y_1 - y_2| > 0$ or both
$|x_1 - x_2| + |y_1 - y_2| > 0$

$\therefore$d(x,y)> 0

(2) Let $x = (x_1, y_1) \in R^2$.By definition we have
$d(x,x) = |x_1 - x_1| + |y_1 - y_1|$ ∴d(x,x)= 0

(3) For $x = (x_1, y_1), y = (x_2, y_2) \in R^2$, we have
$d(x,y) = |x_1 - x_2| + |y_1 - y_2|$
$d(x,y) = |x_2 - x_1| + |y_2 - y_1|$
∴d(x,y)=d(y,x)

(4) For $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3) \in R^2$. We have;
$d(x,y) = |x_1 - x_2| + |y_1 - y_2|$
$d(x,y) = |x_1 - x_3 + x_3 - x_2| + |y_1 - y_3 + y_3 - y_2|$
$d(x,y) \leq |x_1 - x_3| + |x_3 - x_2| + |y_1 - y_3| + |y_3 - y_2|$
$d(x,y) \leq (|x_1 - x_3| + |y_1 - y_3|) + (+|x_3 - x_2| + |y_3 - y_2|)$
$d(x,y) \leq d(x,z) + d(z,y)$
By (1),(2),(3) and (4), we get
d is metric for set R^2.

Examples

Example 4: Define $d : R^2 \times R^2 \rightarrow [0, \infty)$ as
$d(x, y) = Max(|x_1 - x_2|, |y_1 - y_2|)$
where $x = (x_1, y_1)$ and $y = (x_2, y_2)$ are in R^2.
Then show that d is metric for R^2.

Solution: Here function $d : R^2 \times R^2 \rightarrow [0, \infty)$ as
$d(x, y) = Max(|x_1 - x_2|, |y_1 - y_2|)$
where $x = (x_1, y_1)$ and $y = (x_2, y_2)$ are in R^2.

(1) For $x, y \in R^2$. Let $x \neq y$; then $(x_1, y_1) \neq (x_2, y_2)$
$\therefore$ *either* $x_1 \neq x_2$ or $y_1 \neq y_2$ or both
either $|x_1 - x_2| > 0$ or $|y_1 - y_2| > 0$ or both
$Max(|x_1 - x_2|, |y_1 - y_2|) > 0$

$\therefore$d(x,y)> 0

(2) Let $x = (x_1, y_1) \in R^2$.By definition we have
$d(x, x) = Max(|x_1 - x_1|, |y_1 - y_1|) = Max(0, 0)$
$\therefore$d(x,x)= 0

3) For $x = (x_1, y_1), y = (x_2, y_2) \in R^2$, we have
$d(x, y) = Max(|x_1 - x_2|, |y_1 - y_2|)$
$d(x, y) = Max(|x_2 - x_1|, |y_2 - y_1|)$
$\therefore$d(x,y)=d(y,x)

(4) For $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3) \in R^2$. We have;

$d(x, y) = Max(|x_1 - x_2|, |y_1 - y_2|)$

Now,

$|x_1 - x_2| = |x_1 - x_3 + x_3 - x_2| \leq |x_1 - x_3| + |x_3 - x_2|$

Similarly,

$|y_1 - y_2| = |y_1 - y_3 + y_3 - y_2| \leq |y_1 - y_3| + |y_3 - y_2|$

$|x_1 - x_2| \leq Max(|x_1 - x_3|, |y_1 - y_3|) + Max(|x_3 - x_2|, |y_3 - y_2|)$

Similarly, $|y_1 - y_2| \leq Max(|x_1 - x_3|, |y_1 - y_3|) + Max(|x_3 - x_2|, |y_3 - y_2|)$

$Max(|x_1 - x_2|, |y_1 - y_2|) \leq Max(|x_1 - x_3|, |y_1 - y_3|) + Max(|x_3 - x_2|, |y_3 - y_2|)$

$d(x, y) \leq d(x, z) + d(z, y)$

By (1),(2),(3) and (4), we get

d is metric for set R^2.

Examples

Example 5: Define $d : R^n \times R^n \to [0, \infty)$ as

$$d(x, y) = \left[\sum_{k=1}^{n}(x_k - y_k)^2\right]^{1/2}$$

where $x = (x_1, x_2, .., x_n)$, and $y = (y_1, y_2, .., y_n)$ are in R^n. Then show that d is metric for R^n.

Solution: Here function

$$d(x, y) = \left[\sum_{k=1}^{n}(x_k - y_k)^2\right]^{1/2}$$

where $x = (x_1, x_2, .., x_n)$, and $y = (y_1, y_2, .., y_n)$ are in R^n.

(1) For $x, y \in R^n$. Let $x \neq y$; then

$$(x_1, x_2, .., x_n) \neq (y_1, y_2, .., y_n)$$

$$\implies x_k \neq y_k \text{ for some } k$$

$$\implies (x_k - y_k)^2 > 0$$

$$\implies \sum_{k=1}^{n} (x_k - y_k)^2 > 0$$

$$\implies \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{1/2} > 0$$

$$\implies d(x,y) > 0$$

(2) Let $x = (x_1, x_2, ..., x_n) \in R^n$.By definition we have

$$d(x,x) = \left[\sum_{k=1}^{n}(x_k - x_k)^2\right]^{1/2} \text{ for some } k$$

$$d(x,x) = \left[\sum_{k=1}^{n}(0 - 0)^2\right]^{1/2}$$

$\therefore$d(x,x)= 0

3) For $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R^n$, we have

$$d(x,y) = \left[\sum_{k=1}^{n}(x_k - y_k)^2\right]^{1/2}$$

$$d(x,y) = \left[\sum_{k=1}^{n}(y_k - x_k)^2\right]^{1/2}$$

$\therefore$d(x,y)=d(y,x)

(4) Let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) and z = (z_1, z_2, ..., z_n)$ be the points in R^n

$$d(x,y) = \left[\sum_{k=1}^{n}(x_k - y_k)^2\right]^{1/2}$$

$$d(x,y) = \left[\sum_{k=1}^{n}[(x_k - z_k) + (z_k - y_k)]^2\right]^{1/2}$$

$$d(x,y) = \left[\sum_{k=1}^{n}(a_k + b_k)^2\right]^{1/2}$$ where

$a_k = (x_k - z_k) b_k = (z_k - y_k)$

By Minkowiski Inequality we get,

$$d(x,y) \leq \left[\sum_{k=1}^{n}(a_k)^2\right]^{1/2} + \left[\sum_{k=1}^{n}(b_k)^2\right]^{1/2}$$

$d(x,y) \leq d(x,z) + d(z,y)$

By (1),(2),(3) and (4), we get d is metric for set R^2.