# Lecture No-3: Metric Spaces

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# Contents

1. Examples.
2. Definition of Standard matrices.

# Problems



1. Show that $\bar{d}$ is a metric on $C[a, b]$, where

$$\bar{d}(x, y) = \int_a^b |x(t) - y(t)| dt.$$

2. Show that the discrete metric is a metric.
3. Sequence space $s$: set of all sequences of complex numbers with the metric

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}. \quad (1)$$

# Solution

## Solution.

1. $\bar{d}(x,y)=0 \iff$ —x(t)-y(t)—=0 $forall$t $\in [a,b]$ because of the continuity. We have $\bar{d}(x,y)\geq 0$ and $\bar{d}(x,y)=\bar{d}(y,x)$ trivially. We can argue the triangle inequality as follows::

$$\bar{d}(x,y)=\int_a^b |x(t)-y(t)|dt \leq \int_a^b |x(t)-z(t)|dt+\int_a^b$$

□

# Solution

## Solution.

Left as an exercise.

We show only the triangle inequality. Let $a, b \in R$. Then we have the inequalities

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|},$$

where in the first step we have used the monotonicity of the function

$$f(x) = \frac{x}{1+x} = 1 - \frac{1}{1+x}, \text{ for } x > 0.$$

# Standard Metrics



(1) **Euclidean metric on $\mathbb{R}^n$:** The usual Euclidean norm gives a metric on $\mathbb{R}^n$.

$$d(x, y) = ||x - y|| = \left[\sum_{j=1}^{n} |x_j - y_j|^2\right]^{1/2}$$

# Standard Metrics

(2) $l^p$ **norm on** $\mathbb{R}^n$**:** There are other norms we can put on $\mathbb{R}^n$ and hence other metrics. For $1 \leq p < \infty$, define

$$||x||_p = \left[\sum_{j=1}^{n} |x_j|^p\right]^{1/p}$$

(The case $p = 2$ is the usual Euclidean metric.) It is not hard to show that we get the same collection of open sets, i.e., the same topology, for all the value of $p$. As $p \to \infty$ we get:

# Standard Metrics

(3) **sup norm on $\mathbb{R}^n$:** The function

$$d(x, y) = \max_{1 \leq j \leq n} |x_j - y_j|$$

is another metric on $\mathbb{R}^n$ that defines the same topology.

# Standard Metrics



(4) $l^p(\mathbb{N})$**:** If we look at infinite sequences instead of just vectors, things are more interesting. Let $l^p$ be the set of sequences $(x_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$. For such a sequence we define

$$||(x_n)_{n=1}^{\infty}||_p = \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p}$$

Consider the two sets

$$F = \{(x_n)_{n=1}^{\infty} \in l^p : x_n \geq 0 \,\forall n\}$$
$$U = \{(x_n)_{n=1}^{\infty} \in l^p : x_n > 0 \,\forall n\}$$

Is $F$ closed? Is $U$ open?

# Standard Metrics

(5) $l^{\infty}(\mathbb{N})$**:** The space is now the set of bounded infinite sequences. The norm is

$$||(x_n)_{n=1}^{\infty}||_{\infty} = \sup_{1 \leq n < \infty} |x_n|$$

Notice that some of these have the same convergence properties. Suppose $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^n$ converges to $x_\infty$ in the $p$-norm, i.e., $\|x_k - x_\infty\|_p \to 0$ as $k \to \infty$. Note that

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty .$$

It follows that for any $p$ and $q$,

$$n^{-1/q} \|x\|_q \leq \|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty \leq n^{1/p} \|x\|_q$$

so we have that the $p$ and $q$ norms are equivalent. It follows that a sequence converges in $p$-norm if and only if it converges in $q$-norm. So "convergence" is not just a norm property, but something more general. The same can be said for equivalent metric spaces. So what is the most general object for which convergence makes sense?