Lecture No-4: Metric Spaces

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Definition of Normed Linear Space

My Inspiration
Shri. V.G. Patil Saheb
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Definition

Let V be a vector space over $\mathbb{F}$ (with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and $N : V \to \mathbb{R}$ a map such that, writing $N(\mathrm{u}) = \|\mathrm{u}\|$, the following results hold.

(i) $\|\mathrm{u}\| \geq 0$ for all $\mathrm{u} \in V$.

(ii) If $\|\mathrm{u}\| = 0$, then $\mathrm{u} = 0$.

(iii) If $\lambda \in \mathbb{F}$ and $\mathrm{u} \in V$, then $\|\lambda \mathrm{u}\| = |\lambda| \|\mathrm{u}\|$.

(iv) [Triangle law.] If $\mathrm{u}, \mathrm{v} \in V$, then $\|\mathrm{u}\| + \|\mathrm{v}\| \geq \|\mathrm{u} + \mathrm{v}\|$.

Then we call $\| \ \|$ a *norm* and say that $(V, \| \ \|)$ is a *normed vector space*.

Theorem

Any normed vector space can be made into a metric space in a natural way.

Theorem

If $(V, \| \ \|)$ is a normed vector space, then the condition

$$d(\mathrm{u}, \mathrm{v}) = \|\mathrm{u} - \mathrm{v}\|$$

defines a metric d on V.

Proof

Proof.

We observe that

$$d(\mathrm{u}, \mathrm{v}) = \|\mathrm{u} - \mathrm{v}\| \geq 0$$

and

$$d(\mathrm{u}, \mathrm{u}) = \|0\| = \|00\| = |0|\|0\| = 0\|0\| = 0.$$

Further, if $d(\mathrm{u}, \mathrm{v}) = 0$, then $\|\mathrm{u} - \mathrm{v}\| = 0$ so $\mathrm{u} - \mathrm{v} = 0$ and $\square$

Proof

Proof.

u = v. We also observe that

$$d(\mathrm{u},\mathrm{v}) = \|\mathrm{u}-\mathrm{v}\| = \|(-1)(\mathrm{v}-\mathrm{u})\| = |-1|\|\mathrm{v}-\mathrm{u}\| = d(\mathrm{v},\mathrm{u})$$

and

$$\begin{aligned} d(\mathrm{u},\mathrm{v}) + d(\mathrm{v},\mathrm{w}) &= \|\mathrm{u}-\mathrm{v}\| + \|\mathrm{v}-\mathrm{w}\| \\ &\geq \|(\mathrm{u}-\mathrm{v}) + (\mathrm{v}-\mathrm{w})\| \\ &= \|\mathrm{u}-\mathrm{w}\| = d(\mathrm{u},\mathrm{w}). \end{aligned}$$

$\Longrightarrow$ d is metric on V. vm

□

Theorem

Suppose that $a < b$ and we consider the space $C([a, b])$ of continuous functions $f : [a, b] \to \mathbb{R}$ made into a vector space in the usual way.
(i) The equation

$$\langle f, g \rangle = \int_a^b f(t)g(t)\, dt$$

defines an inner product on $C([a, b])$. We write

$$\|f\|_2 = \left(\int_a^b f(t)^2\, dt \right)^{1/2}$$

for the derived norm.

Theorem

(ii) The equation

$$\|f\|_1 = \int_a^b |f(t)|\, dt$$

defines a norm on $C([a,b])$. This norm does not obey the parallelogram law.
(iii) The equation

$$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|.$$

defines a norm on $C([a,b])$. This norm does not obey the parallelogram law.

Proof

Proof.

(i) We have

$$\langle f, f \rangle = \int_a^b f(t)^2 \, dt \geq 0.$$

If $\langle f, f \rangle = 0$, then $\int_a^b f(t)^2 \, dt = 0$ and, by Lemma, $f(t)^2 = 0$ for all t so $f(t) = 0$ for all t and $f = 0$. If we take $a = 0$, $b = 1$,

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1/4 \\ 1/2 - t & \text{if } 1/4 \leq t \leq 1/2 \\ 0 & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Proof

Proof.

We have

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,dt = \int_a^b g(t)f(t)\,dt = \langle g, f\rangle$$

$$\langle f+g, h\rangle = \int_a^b (f(t)+g(t))h(t)\,dt$$

$$= \int_a^b f(t)h(t)\,dt + \int_a^b g(t)h(t)\,dt = \langle f, h\rangle + \langle g, h\rangle$$

$$\langle \lambda f, g\rangle = \int_a^b \lambda f(t)g(t)\,dt = \lambda \int_a^b f(t)g(t)\,dt = \lambda\langle f, g\rangle,$$

so we have an inner product. If we take $a = 0$, $b = 1$,

$$\begin{cases} t & \text{if } 0 \le t \le 1/4 \end{cases}$$

Proof

Proof.

(ii) Observe that

$$\|f\|_1 = \int_a^b |f(t)|\, dt \geq 0$$

and that, if $\|f\|_1 = 0$, then

$$\int_a^b |f(t)|\, dt = 0,$$

so, by Lemma, $|f(t)| = 0$ for all t so $f(t) = 0$ for all t and $f = 0$. If we take $a = 0$, $b = 1$,

$$\begin{cases} t & \text{if } 0 \leq t \leq 1/4 \end{cases}$$

Proof

Proof.

Further

$$\|\lambda f\|_1 = \int_a^b |\lambda||f(t)|\,dt = |\lambda| \int_a^b |f(t)|\,dt = |\lambda|\|f\|_1$$

and, since $|f(t) + g(t)| \leq |f(t)| + |g(t)|$, we have

$$\|f+g\|_1 = \int_a^b |f(t)+g(t)|\,dt \leq \int_a^b |f(t)|+|g(t)|\,dt = \|f\|_1$$

so we have a norm. If we take $a = 0$, $b = 1$,

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1/4 \\ 1/2 - t & \text{if } 1/4 \leq t \leq 1/2 \end{cases}$$

Proof

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If we take $a = 0,\ b = 1,$

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and $g(t) = f(1-t)$, then

$$\|f+g\|_1^2 + \|f-g\|_1^2 = (1/8)^2 + (1/8)^2 = 1/32 \neq 2((1/16)^2 + ($$

so the parallelogram equality fails.

□

Proof

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$$\|f+g\|_1^2+\|f-g\|_1^2 = (1/8)^2+(1/8)^2 = 1/32 \neq 2((1/16)^2+($$

so the parallelogram equality fails. (iii) Observe that $|f(t)| \geq 0$ so

$$\|f\|_{\infty} = \sup |f(t)| \geq 0$$

Proof

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