Lecture No-5: Metric Spaces

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1 Metric Spaces

[Metric Space] The pair (X, d), where X is a set and the function

$$d : X \times X \to R$$

is called a metric space if

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

[Metric Spaces]

1. $d(x, y) = |x - y|$ in R.
2. $d(x, y) = [\sum_{i=1}^{n} (x_i - y_i)^2]^{\frac{1}{2}}$ in R^n.
3. $d(x, y) = \|x - y\|$ in a normed space.

4. Let (X, ρ), (Y, σ) be metric spaces and define the Cartesian product $X \times Y = \{(x, y) | x \in X, y \in Y\}$. Then the product measure $\tau((x_1, y_1), (x_2, y_2)) = [\rho(x_1, x_2)^2 + \sigma(y_1, y_2)^2]^{\frac{1}{2}}$.
5. (Subspace) $(Y, \bar{d})$ of (X, d) if $Y \subset X$ and $\bar{d} = d_{|Y \times Y}$.
6. l^{∞}. Let X be the set of all bounded sequences of complex numbers, i.e., $x = (\xi_i)$ and $|\xi_i| \leq c_x, i$. Then
$$d(x, y) = \sup_{i \in N} |\xi_i - \eta_i|$$
defines a metric on X.
7. $X = C[a, b]$ and
$$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|.$$

8 (Discrete metric)

$$d(x,y)=\begin{cases}1 & x=y\\ 0 & x\neq y\end{cases}.$$

9 l^p. $x=(\xi_i)\in l^p$ if $\sum_{i=1}^{\infty}|\xi_i|^p<\infty$, ($p\geq 1$, fixed),

$$d(x,y)=\sum_{i=1}^{\infty}|\xi_i-\eta_i|^{p^{\frac{1}{p}}}.$$

1 Show that $\bar{d}$ is a metric on $C[a,b]$, where

$$\bar{d}(x,y)=\int_a^b|x(t)-y(t)|dt.$$

2 Show that the discrete metric is a metric.

3. Sequence space s: set of all sequences of complex numbers with the metric

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}. \tag{1}$$

Solution.

1. $\bar{d}(x, y) = 0 \iftrue\fi \iff$ —x(t)-y(t)—=0*forall*t $\in [a, b]$ because of the continuity. We have $\bar{d}(x, y) \geq 0$ and $\bar{d}(x, y) = \bar{d}(y, x)$ trivially. We can argue the triangle inequality as follows::

$$\bar{d}(x, y) = \int_a^b |x(t) - y(t)| dt \leq \int_a^b |x(t) - z(t)| dt + \int_a^b$$

2. Left as an exercise.
3. We show only the triangle inequality. Let $a, b \in R$. Then we have the inequalities

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|},$$