Lecture No-5: Metric Spaces

Santosh Shivlal Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

santosh2maths@gmail.com

July 30, 2021

[Open Ball, Closed Ball, Sphere]

1. $B(x_0, r) = \{x \in X | d(x, x_0) < r\}$
2. $\bar{B}(x_0, r) = \{x \in X | d(x, x_0) \leq r\}$
3. $S(x_0, r) = \{x \in X | d(x, x_0) = r\}$

[Open, Closed, Interior]

1. M is open if contains a ball about each of its points.
2. $K \subset X$ is closed if $K^c = X - K$ is open.
3. $B(x_0;)$ denotes the neighborhood of x_0.
4. $Int(M)$ denotes the interior of M.

[Induced Topology] Consider the set X with the collection τ of all open subsets of X. Then we have

1. $\emptyset \in \tau, X \in \tau$.
2. The union of any members of τ is a member of τ.

3. The finite intersection of members of τ is a member of τ.

We call the pair (X, τ) a topological space and τ a topology for X. It follows that a metric space is a topological space.

[Continuous] Let $X = (X, d)$ and $Y = (Y, \bar{d})$ be metric spaces. The mapping $T : X \to Y$ is continuous at $x_0 \in X$ if for every > 0 there is > 0 such that

$$\bar{d}(Tx, Tx_0) < \ , x \text{ such that } d(x, x_0) < .$$

Theorem (Continuous Mapping)

$T : X \to Y$ is continuous if and only if the inverse image of any open subset of Y is an open subset of X.

Proof.

1. Suppose that T is continuous. Let $S \subset Y$ be open S_0 the inverse image of S. Let $S_0 \neq \emptyset$ and take $x_0 \in S_0$. We have $Tx_0 = y_0 \in S$. Since S is open there exists an -neighborhood of y_0, say $N \subset S$ such that $y_0 \in N$. The continuity of T implies that x_0 has a -neighborhood N_0 which is mapped into N. Since $N \subset S$ we get that $N_0 \subset S_0$, and it follows that S_0 is open.
2. Assume that the inverse image of every open set in Y is an open set in X. Then $x_0 \in X$, and N (-neighborhood of Tx_0) the inverse image N_0 of N is open. Therefore N_0 contains a -neighborhood of x_0. Thus T is continuous.

Some more definitions: [Accumulation Point] $x \in M$ is said to be an accumulation point of M if $(x_n) \subset M x_n \to x$. [Closure] M is the closure of M. [Dense Set] $M \subset X$ is in X dense if $M = X$. [Separable Space] X is separable if there is a countable subset which is dense in X.

1. If M is dense, then every ball in X contains a point of M.
2. R, C are separable.
3. A discrete metric space is separable if and only if it is countable.