Lecture No-9: System of Linear Equations and Matrices

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Linear Algebra - I

Unit I : System of Linear Equations and Matrices

Lecture - 9

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My Inspiration Shri. V.G. Patil Saheb
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3 Elementary row operations and their corresponding matrices

As we'll see shortly, each of the 3 elementary row operations can be performed by multiplying the augmented matrix $(A|\mathbf{y})$ on the *left* by what we'll call an *elementary matrix*. Just so this doesn't come as a total shock, let's look at some simple matrix operations:

- Suppose EA is defined, and suppose the first row of E is $(1, 0, 0, \ldots, 0)$. Then the first row of EA is *identical* to the first row of A.
- Similarly, if the i^{th} row of E is all zeros except for a 1 in the i^{th} slot, then the i^{th} row of the product EA is identical to the i^{th} row of A.
- It follows that if we want to *change* only row i of the matrix A, we should multiply A on the left by some matrix E with the following property:

  Every row *except* row i should be the i^{th} row of the corresponding identity matrix.

The procedure that we illustrate below can (and is) used to reduce *any* matrix to echelon form (not just augmented matrices).

Example: Let

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 0 \end{pmatrix}.$$

1. To multiply the first row of A by $1/3$, we can multiply A on the left by the elementary matrix

   $$E_1 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}.$$

   The result is

   $$E_1A = \begin{pmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 2 & -1 & 0 \end{pmatrix}.$$

   You should check this on your own. Same with the remaining computations.

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2. To add -2*row1 to row 2 in the resulting matrix, multiply it by

$$E_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

to obtain

$$E_2E_1A = \begin{pmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & -\frac{11}{3} & -\frac{10}{3} \end{pmatrix}.$$

3. Now multiply row 2 of E_2E_1A by $-3/11$ using the matrix

$$E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{3}{11} \end{pmatrix},$$

yielding

$$E_3E_2E_1A = \begin{pmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & \frac{10}{11} \end{pmatrix}.$$

4. Finally, we clean out the second column by adding (-4/3)row 2 to row 1. We multiply by

$$E_4 = \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{pmatrix}$$

obtaining

$$E_4E_3E_2E_1A = \begin{pmatrix} 1 & 0 & \frac{5}{11} \\ 0 & 1 & \frac{10}{11} \end{pmatrix}.$$

Of course we get the same result as before, so why bother? The answer is that we're in the process of developing an algorithm that will work in the general case. So it's about time to formally identify our goal in the general case. We begin with some definitions.

Definition: The *leading entry* of a matrix row is the first non-zero entry in the row, starting from the left. A row without a leading entry is a row of zeros.

Definition: The matrix R is said to be in *echelon form* provided that

1. The leading entry of every non-zero row is a 1.

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2. If the leading entry of row i is in position k, and the next row is not a row of zeros, then the leading entry of row $i+1$ is in position $k+j$, where $j \geq 1$.

3. All zero rows are at the bottom of the matrix.

The following matrices are in echelon form:

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here the asterisks (*) stand for any number at all, including 0.

Definition: The matrix R is said to be in *reduced echelon form* if (a) R is in echelon form, and (b) each leading entry is the *only* non-zero entry in its column. The reduced echelon form of a matrix is also called the *Gauss Jordan* form.

The following matrices are in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise: Suppose A is 3×5. What is the maximum number of leading 1's that can appear when it's been reduced to echelon form? Same questions for $A_{5\times 3}$. Can you generalize your results to a statement for $A_{m\times n}$?. (State it as a theorem.)

Once a matrix has been brought to echelon form, it can be put into reduced echelon form by cleaning out the non zero entries in any column containing a leading 1. For example, if

$$R = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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which is in echelon form, then it can be reduced to Gauss-Jordan form by adding (-2)row 2 to row 1, and then (-3)row 3 to row 1. Thus

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

and

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that column 3 cannot be "cleaned out" since there's no leading 1 there.

There is one more elementary row operation and corresponding elementary matrix we may need. Suppose we want to reduce the following matrix to Gauss-Jordan form

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 0 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix}.$$

Multiplying row 1 by 1/2, and then adding -row 1 to row 3 leads to

$$E_2E_1A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 & -1 \\ 0 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{-1}{2} \\ 0 & 0 & 3 \\ 0 & -2 & \frac{5}{2} \end{pmatrix}.$$

Now we can clearly do 2 more operations to get a leading 1 in the (2,3) position, and another leading 1 in the (3,2) position. But this won't be in echelon form (why not?) We need to interchange rows 2 and 3. This corresponds to changing the order of the equations, and evidently doesn't change the solutions. We can accomplish this by multiplying on the left with a matrix obtained from I by interchanging rows 2 and 3:

$$E_3E_2E_1A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & \frac{-1}{2} \\ 0 & 0 & 3 \\ 0 & -2 & \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{-1}{2} \\ 0 & -2 & \frac{5}{2} \\ 0 & 0 & 3 \end{pmatrix}.$$

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Exercise: Without doing any further computation, write down the Gauss-Jordan form for this matrix.

Exercise: Use elementary matrices to reduce

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

to Gauss-Jordan form. You should wind up with an expression of the form

$$E_k \cdots E_2 E_1 A = I.$$

What can you say about the matrix $B = E_k \cdots E_2 E_1$?