Lecture No-10: Topology of Metric Spaces

Santosh Shivlal Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

santosh2maths@gmail.com

August 17, 2021

Contents

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

Lecture 10: Metric Spaces

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

Sanjeevan Gramin Vidyakiya & Samajik Sahayata Pratishthan's
Arts,Commerce & Science College,Onde
Tal. Vikramgad,Dist. Palghar (MS)-401605
(Affiliated to the University of Mumbai)
NAAC Accrediated- Grade-C (CGPA-1.85)
ISO-9001:2015 Certified
Year of Establishment: 2002

Topology of Metric spaces

Unit I : Metric Spaces
Lecture- 10

Santosh Shivlal Dhamone
Assistant Professor in Mathematics
Arts Commerce and Science College, Onde

Lecture 10: Metric Spaces

Lecture 10: Metric Spaces

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

Examples of Open sets :

Example 1: Let $X = \mathbb{R}$, d = standard metric.
Let $A = (a, b)$ be an open interval, then A is an open set.

Proof :-

Let $x \in (a, b)$ be any arbitrary point in (a, b)

Let $\varepsilon = \min\{|a - x|, |x - b|\}$
$= \min\{x - a, b - x\}$ as $a < x < b$

We will prove that

$$B(x, \varepsilon) \subseteq (a, b) = A$$

$B(x, \varepsilon) = (x-\varepsilon, x+\varepsilon)$

Let $y \in B(x,\varepsilon) = (x-\varepsilon, x+\varepsilon)$

$\Rightarrow x-\varepsilon < y < x+\varepsilon$

if $\varepsilon \leq x-a \Rightarrow x-\varepsilon \geqslant a$

& if $\varepsilon \leq b-x \Rightarrow x+\varepsilon \geqslant b$

$\therefore a \leq x-\varepsilon < y \leq x+\varepsilon = b$

$\Rightarrow a < y < b$

$\Rightarrow y \in (a,b)$

Thus $y \in B(x,\varepsilon) = (x-\varepsilon, x+\varepsilon) \Rightarrow y \in (a,b)$

$\Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq A = (a,b)$

since the point x is an arbitrary point, we get A is an open set.

Lecture 10: Metric Spaces

My Inspiration Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

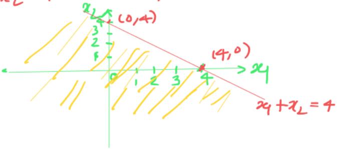
Example 2:- Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ & d = Euclidean metric.
Let $G = \{(x_1, x_2) \mid x_1 + x_2 < 4\}$ then G is open in X.

Solution :- Geometrically, we can sketch the region representing G.
First we sketch the straight line $x_1 + x_2 = 4$

$x_1 + x_2 = 4$

if $x_2 = 0 \Rightarrow x_1 = 4 \quad \therefore (4, 0)$

if $x_1 = 0 \Rightarrow x_2 = 4 \quad \therefore (0, 4)$



My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

The straight line divides the plane R^2 into 2 points one containing origin & other not containing origin.
Substituting $x_1=0$, $x_2=0$ in $x_1+x_2<4$ we get
$0+0<4 \Rightarrow 0<4$ which is true.
Thus origin satisfies the inequality $x_1+x_2<4$.
Hence the region containing origin $(0,0)$ is the required region representing G.

Let $p=(p_1,p_2)$ be an arbitrary point in G.
To obtain open ball around p which lie inside G
we first obtain the perpendicular distance from $p=(p_1,p_2)$ to the line $x_1+x_2=4$ which is obtain by the formula

Formula :-
Perpendicular distance of a point (x_0, y_0) from the line $ax+by+c=0$ is $\left|\dfrac{ax_0+by_0+c}{\sqrt{a^2+b^2}}\right|$

Here $x+y=4 \Rightarrow x+y-4=0$
$\therefore a=1, b=1, c=-4$

point $p=(p_1, p_2)$
Hence perpendicular distance from $p=(p_1, p_2)$ to the line $x_1+x_2-4=0$ is $\left|\dfrac{p_1+p_2-4}{\sqrt{1^2+1^2}}\right| = \left|\dfrac{p_1+p_2-4}{\sqrt{2}}\right|$

Lecture 10: Metric Spaces

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

Note that $(p_1, p_2) \in G$ $\quad \because G = \{(x_1, x_2) \mid x_1 + x_2 < 4\}$

$$\Rightarrow p_1 + p_2 < 4$$
$$\Rightarrow p_1 + p_2 - 4 < 0$$
$$\Rightarrow |p_1 + p_2 - 4| = -p - p_2 + 4$$

Hence perpendicular distance

$$= \frac{4 - p_1 - p_2}{\sqrt{2}}$$

Take $\varepsilon = \dfrac{4 - p_1 - p_2}{2} < \dfrac{4 - p_1 - p_2}{\sqrt{2}}$

Note that for positive values $a, b > 0$

$$(a+b)^2 > a^2 + b^2 \Rightarrow a + b > \sqrt{a^2 + b^2}$$

Similarly $-\varepsilon < y_2 - p_2 < \varepsilon$

$\Rightarrow -2\varepsilon < (y_1+y_2) - (p_1+p_2) < 2\varepsilon$

$\Rightarrow y_1 + y_2 < 2\varepsilon + (p_1+p_2)$

But $\varepsilon = \dfrac{4-p_1-p_2}{2}$

$\therefore y_1+y_2 < \cancel{2} \times \left(\dfrac{4-p_1-p_2}{\cancel{2}}\right) + p_1 + p_2$

$\therefore y_1 + y_2 < 4 - \cancel{p_1} - \cancel{p_2} + \cancel{p_1} + \cancel{p_2}$

$\therefore y_1 + y_2 < 4$

$\Rightarrow y_1 + y_2 \in G$ This implies $B((p_1,p_2),\varepsilon) \subseteq G$.

since (p_1,p_2) is an arbitrary point in G.
We get G is an open set.

$a+b > \sqrt{a^2+b^2}$

$\Rightarrow \frac{1}{a+b} < \frac{1}{\sqrt{a^2+b^2}}$

We will prove that $B((p_1,p_2), \varepsilon) \subseteq G$

Let $(y_1, y_2) \in B((p_1,p_2), \varepsilon) \subseteq G$.

$\Rightarrow d((y_1,y_2),(p_1,p_2)) < \varepsilon$

$\Rightarrow \sqrt{(y_1-p_1)^2+(y_2-p_2)^2} < \varepsilon$

Note that $|y_1-p_1| = \sqrt{|y_1-p_1|^2} \leq \sqrt{|y_1-p_1|^2+|y_2-p_2|^2} < \varepsilon$

$\Rightarrow |y_1-p_1| < \varepsilon \Rightarrow -\varepsilon < y_1-p_1 < \varepsilon$