Practical No 2: Metric Spaces

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Example 1: [Metric Spaces]

1. $d(x, y) = |x - y|$ in R.
2. $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{1}{2}}$ in R^n.
3. $d(x, y) = \|x - y\|$ in a normed space.
4. Let (X, ρ), (Y, σ) be metric spaces and define the Cartesian product $X \times Y = \{(x, y)|x \in X, y \in Y\}$. Then the product measure $\tau((x_1, y_1), (x_2, y_2)) = [\rho(x_1, x_2)^2 + \sigma(y_1, y_2)^2]^{\frac{1}{2}}$.
5. (Subspace) $(Y, \bar{d})$ of (X, d) if $Y \subset X$ and $\bar{d} = d_{|Y \times Y}$.
6. l^{∞}. Let X be the set of all bounded sequences of complex numbers, i.e., $x = (\xi_i)$ and $|\xi_i| \leq c_x, i$. Then

$$d(x, y) = \sup_{i \in N} |\xi_i - \eta_i|$$

Examples

Example 2:

1. Show that $\bar{d}$ is a metric on $C[a, b]$, where

$$\bar{d}(x, y) = \int_a^b |x(t) - y(t)| dt.$$

2. Show that the discrete metric is a metric.
3. Sequence space s: set of all sequences of complex numbers with the metric

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}. \tag{1}$$

Solution:

(1) $\bar{d}(x,y) = 0 \iff$ —x(t)-y(t)—=0$forall$t $\in [a,b]$ because of the continuity. We have $\bar{d}(x,y) \geq 0$ and $\bar{d}(x,y) = \bar{d}(y,x)$ trivially.
We can argue the triangle inequality as follows::

$$\bar{d}(x,y) = \int_a^b |x(t)-y(t)|dt \leq \int_a^b |x(t)-z(t)|dt + \int_a^b$$

(2) Left as an exercise.

(3) We show only the triangle inequality. Let $a, b \in R$. Then we have the inequalities

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|},$$

where in the first step we have used the monotonicity of the function

$$f(x) = \frac{x}{1+x} = 1 - \frac{1}{1+x}, \text{ for } x > 0.$$

Substituting $a = \xi_i - \zeta_i$ and $b = \zeta_i - \eta_i$, where $x = (\xi_i)$, $y = (\eta_i)$, and $z = (\zeta_i)$ we get

$$\frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} \leq \frac{|\xi_i - \zeta_i|}{1 + |\xi_i - \zeta_i|} + \frac{|\zeta_i - \eta_i|}{1 + |\zeta_i - \eta_i|}.$$

If we multiply both sides by $\frac{1}{2^i}$ and sum over from $i = 1$ to ∞ we get the stated result.

Open Sets, Closed Sets

[Open Ball, Closed Ball, Sphere]

1. $B(x_0, r) = \{x \in X | d(x, x_0) < r\}$
2. $\bar{B}(x_0, r) = \{x \in X | d(x, x_0) \leq r\}$
3. $S(x_0, r) = \{x \in X | d(x, x_0) = r\}$

[Open, Closed, Interior]

1. M is open if contains a ball about each of its points.
2. $K \subset X$ is closed if $K^c = X - K$ is open.
3. $B(x_0;)$ denotes the neighborhood of x_0.
4. $Int(M)$ denotes the interior of M.

[Continuous] Let $X = (X, d)$ and $Y = (Y, \bar{d})$ be metric spaces. The mapping $T : X \to Y$ is continuous at $x_0 \in X$ if for every > 0 there is > 0 such that

$$\bar{d}(Tx, Tx_0) < \ , x \text{ such that } d(x, x_0) < .$$

Examples

Example 3: ℓ^{∞} is not separable.

Solution: Let $y = (\eta_i)$ where $\eta_i = 0, 1$. There are uncountably many y's. If we put small balls with radius $\frac{1}{3}$ at the y's they will not intersect. It follows that if $M \subset l^\infty$ is dense in l^∞, then M is uncountable. Therefore l^∞ is not separable.

Examples

Example 4: Show that l^p, $1 \leq p < \infty$ is separable.

Solution: Let M the set of all sequences of the form $x = (\xi_1, \xi_2, ..., \xi_n, 0, 0, ...)$, where n is any positive integer and the ξ_is are rational. M is countable. We argue that M is dense in l^p as follows. Let $y = (\eta_i) \in l^p$ be arbitrary. Then for every > 0 there is an n such that

$$\sum_{i=n+1}^{\infty} |\eta_i|^p < \frac{p}{2}.$$

Since the rationals are dense in R, for each η_i there is a rational ξ_i close to it. Hence there is an $x \in M$ such that

$$\sum_{i=1}^{n} |\eta_i - \xi_i| < \frac{p}{2}.$$

It follows that $d(y, x) < ve$.

Examples

Example 5:

1. A convergent sequence (x_n) in X is bounded and its limit x is unique.
2. If $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.

Solution:

(1) Suppose that $x_n \to x$. Then, taking $= 1$ we can find an N such that $d(x_n, x) < 1$ for all $n > N$. By the triangle inequality we have

$$d(x_n, x) < 1 + \max d(x_1, x), d(x_2, x), ..., d(x_N, x).$$

Therefore (x_n) is bounded. If $x_n \to x$ and $x_n \to z$, then

$$0 \leq d(x, z) \leq d(x_n, x) + d(x_n, z) \to 0 \text{ as } n \to \infty$$

and uniqueness of the limit follows.

(2) We have

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n),$$

and hence

$$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y).$$

Interchanging x_n and x, y_n and y, and multiplying by -1 we get

$$d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y).$$

Combining the two inequalities we get

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \to 0 \text{ as } n \to \infty$$