My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

Practical No-3: Topology of Metric Spaces

Santosh Shivlal Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

santosh2maths@gmail.com

August 13, 2021

Contents

Open Sets of a Metric Spaces

Open Ball in a Metric Space :-

Definition : Open Ball :

Let (X, d) be a metric space. Let $p \in X$ & $\varepsilon > 0$, then the open ball centred at point p & radius ε denoted by $B_d(p, \varepsilon)$ is defined as

$$B_d(p, \varepsilon) = \{x \in X \mid d(x, p) < \varepsilon\}$$

Some times it is simply denoted as $B(p, \varepsilon)$ in (X, d)

Examples :-

(1) Let $X = \mathbb{R}$. Let $p \in \mathbb{R}$ & $\varepsilon > 0$ & d = standard metric or absolute value metric.

Solution:-

$$B(p,\varepsilon) = \{x \in \mathbb{R} \mid d(x,p) < \varepsilon\}$$
$$= \{x \in \mathbb{R} \mid |x-p| < \varepsilon\}$$
$$= \{x \in \mathbb{R} \mid -\varepsilon < x-p < \varepsilon\}$$
$$= \{x \in \mathbb{R} \mid p-\varepsilon < x < p+\varepsilon\}$$

$d(x,y) = |x-y|$ standard metric

$\therefore B(p,\varepsilon) = (p-\varepsilon, p+\varepsilon)$

Thus in $(\mathbb{R}, |\,|)$ open balls are open intervals around the point p.

Practical No-3: Metric Spaces

(2) Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Then define open ball in $\mathbb{R}^2$ with respective following metric

(i) d_1 = Euclidean metric

(ii) d_2 = sum metric or L^1 metric

(iii) d_3 = Supremum metric

Solution :- Let $p = (p_1, p_2) \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ & $\varepsilon > 0$

(i) $B_{d_1}(p, \varepsilon) = B_{d_1}((p_1, p_2), \varepsilon)$

$= \{x \in \mathbb{R}^2 \mid d(x, p) < \varepsilon\}$

$\because x \in \mathbb{R}^2$

$\therefore x = (x_1, x_2)$

$B_{d_1}(p, \varepsilon) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid d((x_1, x_2), (p_1, p_2)) < \varepsilon\}$

Practical No-3: Metric Spaces

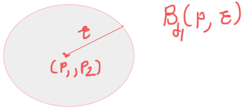
My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

$$\therefore\ B_{d_1}(p,\varepsilon) = \left\{ (x_1,x_2)\in \mathbb{R}^2 \;\middle|\; \sqrt{(x_1-p_1)^2+(x_2-p_2)^2} < \varepsilon \right\}$$

$$= \left\{ (x_1,x_2)\in \mathbb{R}^2 \;\middle|\; (x_1-p_1)^2+(x_2-p_2)^2 < \varepsilon \right\}$$

= Set of all points inside a circle with center at (p_1, p_2) & radius ε.



(ii) Let $X = \mathbb{R}^2$ & $d_2 =$ Sum metric or L^1 metric

$B_{d_2}(p,\varepsilon) = B_{d_2}((p_1,p_2),\varepsilon)$ $\quad \because p \in \mathbb{R}^2 \Rightarrow p = (p_1, p_2)$

$= \{x \in \mathbb{R}^2 \mid d_2(x,p) < \varepsilon\}$

$= \{x \in \mathbb{R}^2 \mid |x_1-p_1| + |x_2-p_2| < \varepsilon\}$ $\quad \because x = (x_1,x_2) \in \mathbb{R}^2$

Note that $\underline{|x_1-p_1|} + \underline{|x_2-p_2|} < \varepsilon$ gives rise to 4 inequalities

(i) $|x_1-p_1|$ & $|x_2-p_2|$ both are positive

$x_1 - p_1 + x_2 - p_2 < \varepsilon$

$\Rightarrow x_1 + x_2 < p_1 + p_2 + \varepsilon$

(ii) $|x_1 - p_1|$ is negative & $|x_2 - p_2|$ is positive

$\therefore -(x_1 - p_1) + (x_2 - p_2) < \varepsilon$

$\Rightarrow -x_1 + x_2 < p_2 - p_1 + \varepsilon$

(iii) $|x_1 - p_1|$ is positive & $|x_2 - p_2|$ is negative

$\therefore (x_1 - p_1) - (x_2 - p_2) < \varepsilon$

$\Rightarrow x_1 - x_2 < p_1 - p_2 + \varepsilon$

(iv) Both $|x_1 - p_1|$ & $|x_2 - p_2|$ are negative

$\therefore -(x_1 - p_1) - (x_2 - p_2) < \varepsilon$

$\Rightarrow -x_1 - x_2 < \varepsilon - p_1 - p_2$

Practical No-3: Metric Spaces

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

$\therefore$ $B_{d_2}(p, \varepsilon)$ is a region $\mathbb{R}^2$ bounded by the straight lines

(i) $x_1 + x_2 = p_1 + p_2 + \varepsilon$

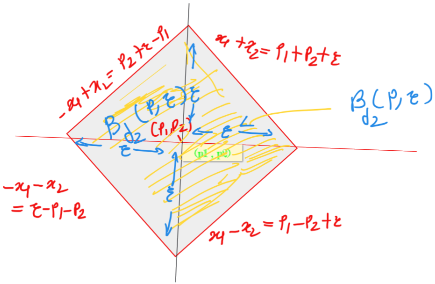
(ii) $-x_1 + x_2 = p_2 - p_1 + \varepsilon$

(iii) $x_1 - x_2 = p_1 - p_2 + \varepsilon$

(iv) $-x_1 - x_2 = \varepsilon - p_1 - p_2$

It can be seen that straight lines represented by (i) & (iv) are parallel & straight lines represented by (ii) & (iii) are parallel & hence $B_{d_2}(p, \varepsilon)$ is a parallelogram with (p_1, p_2) at the center (intersecting point of diagonals) with the equation of boundaries of parallelogram represented by equations (i), (ii), (iii) & (iv).

Practical No-3: Metric Spaces



Practical No-3: Metric Spaces

Open Ball :-

Let (X,d) be a metric space. Let $p \in X$ & $\varepsilon > 0$, then the ball centered at p & radius ε denoted by $B_d(p,\varepsilon)$ or $B(p,\varepsilon)$ is defined as

$$B_d(p,\varepsilon) = \{ x \in X \mid d(x,p) < \varepsilon \}$$

Examples :-

(4) Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ & d = Supremum metric.
Then find $B_d(p, \varepsilon)$

Solution :-

Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ &
d = supremum metric.
$\therefore$ For $x, y \in X = \mathbb{R}^2$, $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$
$\because x = (x_1, x_2)$, $y = (y_1, y_2)$
Here we have to find $B(p, \varepsilon)$ w.r.t. supremum metric
$\therefore B(p, \varepsilon) = \{x \in \mathbb{R}^2 \mid d(x, p) < \varepsilon\}$

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

$$B(p,\varepsilon) = \{ x \in \mathbb{R}^2 \mid d(x,p) < \varepsilon \}$$
$$= \{ (x_1,x_2) \in \mathbb{R}^2 \mid \max\{ |x_1 - p_1|, |x_2 - p_2| \} < \varepsilon \}$$

$\because X = \mathbb{R}^2 \Rightarrow x = (x_1, x_2)$ & $p = (p_1, p_2)$

Note that there are two possibilities either $|x_1 - p_1| < \varepsilon$ or $|x_2 - p_2| < \varepsilon$ which gives rise to 4 equations of boundaries of $B(p,\varepsilon)$ which are follows

if $|x_1 - p_1| < \varepsilon$ (i) $x_1 - p_1 = \varepsilon \Rightarrow x_1 = p_1 + \varepsilon$

(ii) $-x_1 + p_1 = \varepsilon \Rightarrow x_1 = p_1 - \varepsilon$

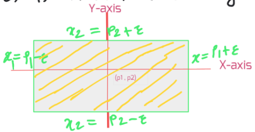
if $|x_2 - p_2| < \varepsilon$ then

(iii) $x_2 - p_2 = \varepsilon \Rightarrow x_2 = p_2 + \varepsilon$

(iv) $-x_2 + p_2 = \varepsilon \Rightarrow x_2 = p_2 - \varepsilon$

We can see that (i) & (ii) represent straight lines parallel to y-axis and equations (iii) & (iv) represents straight lines parallel to x-axis.

$\therefore$ $B(p, \varepsilon)$ is in fact a rectangle.



Practical No-3: Metric Spaces

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

(5) Let X be any non-empty set & d be the discrete metric where

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \quad \text{for } x, y \in X$$

Then find $B(p, \varepsilon)$ w.r.t. discrete metric.

Solution:- Given X is a non-empty set & d is discrete metric defines as

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

We have to find $B(p, \varepsilon)$

Practical No-3: Metric Spaces

Let $0 < \varepsilon < 1$ & $p \in X$

$$B(p, \varepsilon) = \{x \in X \mid d(p, x) < \varepsilon\}$$
$$= \{p\}$$

as the only values taken by $d(x, y) = 0$ or 1.

If $\varepsilon > 1$, $p \in X$

$$B(p, \varepsilon) = \{x \in X \mid d(x, p) < \varepsilon\}$$
$$= X$$

as every pair of points (x, y) in X are at a distance 1 if $x \neq y$

Thus in a discrete metric space, open balls are either singleton sets or X.

Practical No-3: Metric Spaces

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

(6) Let $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$

define $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = \sqrt{9(x_1 - y_1)^2 + 16(x_2 - y_2)^2}.$$

Find $B((0,0), 1)$.

Solution :– By defination of open ball about point p with radius ε is

$$B(p, \varepsilon) = \{ x \in X \mid d(x, p) < \varepsilon \}$$

Here $p = (0,0)$ & $\varepsilon = 1$

$$\therefore B((0,0), 1) = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid d((x_1, x_2), (0,0)) < 1 \}$$

Practical No-3: Metric Spaces

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

$$\therefore B((0,0),1) = \{(x_1,x_2) \in \mathbb{R}^2 \mid d((x_1,x_2),(0,0)) < 1\}$$

$$= \{(x_1,x_2) \in \mathbb{R}^2 \mid \sqrt{9(x_1-0)^2+16(x_2-0)^2} < 1\}$$

$$\because d(x,y) = \sqrt{9(x_1-y_1)^2+16(x_2-y_2)^2}$$

$$= \{(x_1,x_2) \in \mathbb{R}^2 \mid \sqrt{9x_1^2+16x_2^2} < 1\}$$

$$= \{(x_1,x_2) \in \mathbb{R}^2 \mid 9x_1^2+16x_2^2 < 1\}$$

$$= \{(x_1,x_2) \in \mathbb{R}^2 \mid \frac{x_1^2}{1/9} + \frac{x_2^2}{1/16} < 1\}$$

$$= \{(x_1,x_2) \in \mathbb{R}^2 \mid \frac{x_1^2}{(\frac{1}{3})^2} + \frac{x_2^2}{(\frac{1}{4})^2} < 1\}$$

Practical No-3: Metric Spaces

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

$$B((0,0),1) = \left\{ (x_1,x_2) \in \mathbb{R}^2 \;\middle|\; \frac{x_1^2}{(\frac{1}{3})^2} + \frac{x_2^2}{(\frac{1}{4})^2} < 1 \right\}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$B((0,0),1)$ = Interior of ellipse $\frac{x_1^2}{(\frac{1}{3})^2} + \frac{x_2^2}{(\frac{1}{4})^2} = 1$

with center $(0,0)$ & major axis $= a = \frac{1}{3}$

minor axis $= b = \frac{1}{4}$.