My Inspiration
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Lecture No-13: System of Linear Equations and Matrices

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Linear Algebra - I

Unit I : System of Linear Equations and Matrices

Lecture - 13

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Theorem 2.1

Theorem. Let A be a matrix of size $m \times n$. Let E be an elementary matrix (of size $m \times m$) obtained by performing an elementary row operation on I_m and B be the matrix obtained from A by performing the same operation on A. Then $B = EA$.

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Proof.

We will prove only for one operation (out of three) and when when $n = m = 3$. Suppose E is the matrix obtained by interchanging first and third rows.

$$Then, \quad E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad also\ write \quad A = \begin{bmatrix} x & y & z \\ a & b & c \\ u & v & w \end{bmatrix}$$

$$So, \quad EA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & y & z \\ a & b & c \\ u & v & w \end{bmatrix} = \begin{bmatrix} u & v & w \\ a & b & c \\ x & y & z \end{bmatrix}$$

which is obtained by switching first and third rows of A. ■

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Example 2.4.4

Let

$$A = \begin{bmatrix} 1 & 7 & 1 & 17 \\ -1 & 1 & 1 & 8 \\ 8 & 18 & 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 18 & 0 & 9 \\ -1 & 1 & 1 & 8 \\ 1 & 7 & 1 & 17 \end{bmatrix}$$

Find an elementary matrix E so that $B = EA$.

Solution: The matrix B is obtained by switching first and the last row of A. They have size 3×4. By the theorem above, E is obtained by switching first and the last row of I_3. So,

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; so \;\; B = EA \;\; (\text{Directly Check, as well.}).$$

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Example 2.4.5

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 8 \\ 8 & 18 & 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 3 & 10 \\ 8 & 18 & 0 & 9 \end{bmatrix}$$

Find an elementary matrix E so that $B = EA$.

Solution: The matrix B is obtained by adding 2 times the first row of A to the second row of A. By the thorem above, E is obtained from I_3 by adding 2 times its first row to second. So,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ so \quad B = EA \quad \text{(Directly Check, as well.)}.$$

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Example 2.4.6

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 8 \\ 8 & 18 & 0 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 9 & 3 & 3 & 24 \\ 8 & 18 & 0 & 9 \end{bmatrix}$$

Find an elementary matrix E so that $B = EA$.

Solution: The matrix B is obtained from A by multiplying its second row by 3. So, by the theorem E is obtained by doing the same to I_3. So

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad so \quad B = EA \quad \text{(Directly Check, as well.)}.$$

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Definition

Definition. Two matrices A, B of size $m \times n$ are said to be **row-equivalent** if

$$B = E_k E_{k-1} \cdots E_2 E_1 A \quad \textit{where} \quad E_i \textit{ are elemetary}.$$

This is same as saying that B is obtained from A by application of a series of elemetary row operations.

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Theorem 2.2

Theorem. A square matrix A is invertible if and only if it is product of elementary matrices.

Proof. Need to prove two statements. First prove, if A is product it of elementary matrices, then A is invertible. So, suppose $A = E_k E_{k-1} \cdots E_2 E_1$ where E_i are elementary. Since elementary matrices are invertible, E_i^{-1} exists. Write $B = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$. Then

$$AB = (E_k E_{k-1} \cdots E_2 E_1)(E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}) = I.$$

Similarly, $BA = I$. So, B is the inverse of A.

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Proof of "only if":

Conversely, assume A is invertible. We have to prove that A is product of elementary matrices. Since A is invertible. The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. So, the augmented matrix $[A|\mathbf{0}]$ reduces to $[I|\mathbf{0}]$ by application of elementary row operations. So, $E_kE_{k-1}\cdots E_2E_1[A|\mathbf{0}] = [I|\mathbf{0}]$ where E_i are elementary. So

$$E_kE_{k-1}\cdots E_2E_1A = I \quad \text{or} \qquad A = E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}$$

All the factors on the right are elementary. So, A is product of elementary matrices. The proof is complete. ■