Lecture No-17: Topology of Metric Spaces
Unit II : Sequences and Complete Metric Space

Santosh Shivlal Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

santosh2maths@gmail.com

Contents

Lecture 1: Sequences and Complete Metric Space

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

Sanjeevan Gramin Vidyakiya & Samajik Sahayata Pratishthan's
Arts,Commerce & Science College,Onde
Tal. Vikramgad,Dist. Palghar (MS)-401605
(Affiliated to the University of Mumbai)
NAAC Accredited- Grade-C (CGPA-1.85)
ISO-9001:2015 Certified
Year of Establishment: 2002

Topology of Metric Spaces

Unit II : Sequences and Complete metric spaces

Lecture : 1

SANTOSH SHIVLAL DHAMONE

Assistant Professor in Mathematics
Arts Commerce and Science College, Onde
Tal: Vikramgad, Dist: Palghar

Contact Details:
Email ID:
santosh2maths@gmail.com

ssdhamone@acscollegeonde.ac.in
Mobile No:-
9422291033

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

Closed Balls and Closed Sets:

Closed Ball :-

Let (X, d) be a metric space. Let $p \in X$ be any point in X. Let $\varepsilon > 0$ be a real number. The closed ball centered at p and radius ε denoted by $B[p, \varepsilon]$ is defined as :

$$B[p, \varepsilon] = \{ x \in X \mid d(x, p) \leq \varepsilon \}$$

In open ball :

$$B(p, \varepsilon) = \{ x \in X \mid d(x, p) < \varepsilon \}$$

Examples :

(1) Let $X = \mathbb{R}$, d is usual metric or standard metric $p = 2 \in \mathbb{R}$ & $\varepsilon = 1$. Find $B[2, 1]$.

Solution :- Given $X = \mathbb{R}$, d = standard or usual metric

i.e. $\forall\, x, y \in \mathbb{R}$, $d(x, y) = |x - y|$

$p = 2$ & $\varepsilon = 1$

We know that

$$B[p, \varepsilon] = \{x \in X \mid d(x, p) \leq \varepsilon\}$$

$$\therefore B[2, 1] = \{x \in R \mid d(x, 2) \leq 1\}$$

$$B[2, 1] = \{x \in R \mid |x - 2| \leq 1\}$$

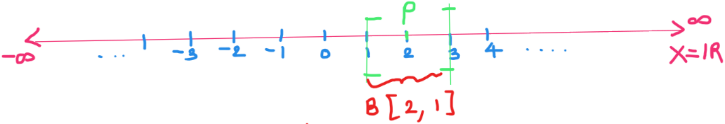
to be continue

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

$$B[2,1] = \{x \in \mathbb{R} \mid |x-2| \leq 1\}$$
$$= \{x \in \mathbb{R} \mid -1 \leq x-2 \leq 1\}$$
$$= \{x \in \mathbb{R} \mid -1+2 \leq x \leq 1+2\}$$
$$= \{x \in \mathbb{R} \mid 1 \leq x \leq 3\}$$

$$\therefore \; B[2,1] = [1, 3]$$



Note :- $B(2,1) = (1,3)$

Lecture 1: Sequences and Complete Metric Space

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

(2) Let $X = \mathbb{R}^2$. Let d = Euclidean metric.
Let $p = (p_1, p_2) = (0, 0)$ & $\varepsilon > 0$ be any real number.
Find $B[p, \varepsilon]$.

Solution :- Given $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$,
d = Euclidean metric $= \sqrt{(x_1 - x_2)^2 + (y_2 - y_1)^2} \quad \forall\, x, y \in X$
$p = (0, 0) = (p_1, p_2)$

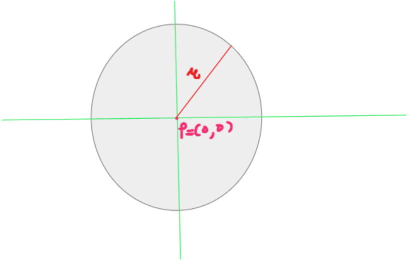
We know that
$$B[p, \varepsilon] = \{ x \in X \mid d(x, p) \leq \varepsilon \}$$

$$\therefore \quad B[(0,0), \varepsilon] = \{ (x, y) \in \mathbb{R}^2 \mid d((x, y), (p_1, p_2)) \leq \varepsilon \}$$
$$= \{ (x, y) \in \mathbb{R}^2 \mid d[(x, y), (0, 0)] \leq \varepsilon \}$$

$$B[(0,0),\varepsilon] = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{(x-0)^2+(y-0)^2} \leq \varepsilon\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \leq \varepsilon\}$$

$$B[(0,0),\varepsilon] = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq \varepsilon^2\}$$



(3) Let X = Any non-empty set, d = discrete metric space.
Find $B[p, \varepsilon]$.

Solution :- Given X is any non-empty set.
d = discrete metric space
i.e. For $x, y \in X$

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Let $p \in X$ be any point.
Let $0 < \varepsilon < 1$.

$$\therefore B[p,1] = \{x \in X \mid d(x,p) \leq \varepsilon < 1\} = \{p\}$$

~~where $\varepsilon < 1$~~

Lecture 1: Sequences and Complete Metric Space

My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

Santosh Shivlal Dhamone

For $\varepsilon = 1$

$$B[p, \varepsilon] = \{x \in X \mid d(x, p) \leq 1\}$$

$$B[p, \varepsilon] = X$$

When $\varepsilon > 1$

$$B[p, \varepsilon] = \{x \in X \mid d(x, p) \leq 1\}$$

$$= X$$

Thus in a discrete metric space only closed balls are a singltone sets & whole of X.

Note that $\{p\}$ is also open ball & X is also open ball

Hence $\{p\}$ & X are both open balls & closed ball.