Lecture No 18: Topology of Metric Spaces

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Lecture 2 : Unit II : Sequences Complete Metric Space

Sanjeevan Gramin Vidyakiya & Samajik Sahayata Pratishthan's
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Topology of Metric Spaces

Unit II : Sequences and Complete Metric Space

Lecture - 2

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Open Sets of Metric Space :-

Let (X, d) be a metric space. Let $G \subseteq X$. G is said to be an open set if for every $p \in G$ there exist a positive real number $\varepsilon > 0$ such that

$$B(p, \varepsilon) \subseteq G.$$

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Dr. V. S. Sonawne

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Closed Set :-

Let (X,d) be a metric space. Let $F \subseteq X$. F is said to be a closed set if $X-F$ is open.

That is, F is closed set if and only if its complement $F^c = X - F$ is open in X.

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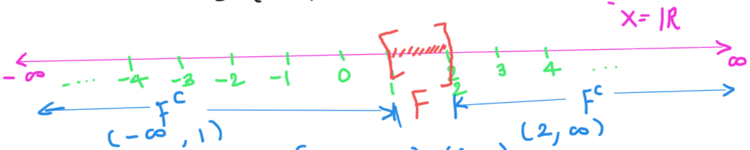
Examples :-

(1) Let $X = \mathbb{R}$, $F = [1, 2]$. Then show that F is closed set in X.

Solution :- Let $X = \mathbb{R}$, $F = [1, 2]$, then

$$F^c = \mathbb{R} - F$$
$$= (-\infty, 1) \cup (2, \infty)$$



$\therefore F^c = (-\infty, 1) \cup (2, \infty)$

To show that F is closed to set in $\mathbb{R}$.

We have to show that $F^c = \mathbb{R} - F$ is open in $\mathbb{R}$.

(i) To show that $(-\infty, 1)$ is open :

For each $p \in (-\infty, 1)$. Take $\varepsilon = 1 - p$

$$B(p, \varepsilon) = (p - \varepsilon, p + \varepsilon)$$
$$= \{ x \in \mathbb{R} \mid |x - p| < \varepsilon \}$$
$$= (p - (1-p), p + (1-p))$$
$$B(p, \varepsilon) = (2p - 1, 1) \subseteq (-\infty, 1)$$

$\therefore (-\infty, 1)$ is open.

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(ii) To show that $(2,\infty)$ is open :-

For every $p \in (2,\infty)$. Take $\varepsilon = p-2$

$$B(p,\varepsilon) = \{x \in \mathbb{R} \mid |x-p| < \varepsilon\}$$
$$= (p-\varepsilon,\ p+\varepsilon) \qquad (\because \text{open ball is an open interval in } \mathbb{R})$$
$$= (p-(p-2),\ p+p-2)$$
$$B(p,\varepsilon) = (2,\ 2p-2) \subseteq (2,\infty)$$

$\therefore (2,\infty)$ is open in $\mathbb{R}$.

Hence $(-\infty,1)$ & $(2,\infty)$ are open sets in $\mathbb{R}$.

$\Rightarrow (-\infty,1) \cup (2,\infty)$ is also open set $(\because$ Union of open sets is open$)$

$\Rightarrow F^c = (-\infty,1) \cup (2,\infty)$ is open in $\mathbb{R}$.

$\Rightarrow F$ is closed in $\mathbb{R}$.

(2) $X = \mathbb{R}^2$. Is $F \subseteq X = \mathbb{R}^2$. Show that F is closed.

Solution :- Given $X = \mathbb{R}^2$, d = Euclidean metric.

For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbb{R}^2$.

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$F = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), (0, 0)) \leq 1\}$$

$$= \{(x, y) \in \mathbb{R}^2 \mid \sqrt{(x-0)^2 + (y-0)^2} \leq 1\}$$

$$F = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

To show that F is closed

i.e. we have to show that F^c is open.

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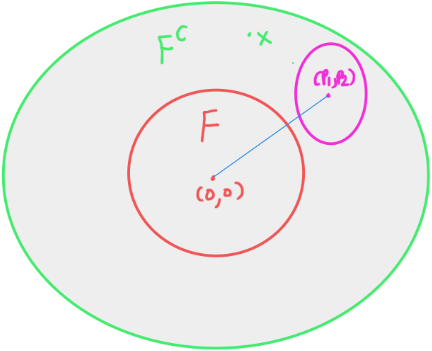

$$F = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}$$

$$F^c = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 1 \}$$

Let $p = (p_1, p_2) \in F^c$

$$d(p, 0) = d((p_1, p_2), (0, 0))$$

$$= \sqrt{(p_1 - 0)^2 + (p_2 - 0)^2}$$

$$= \sqrt{p_1^2 + p_2^2} > 1$$

Let $\varepsilon = \sqrt{p_1^2 + p_2^2} - 1 = d(p, 0) - 1$

We will prove that $B(p, \varepsilon) \subseteq F^c$

Let $x \in B(p, \varepsilon) \Rightarrow d(x, p) < \varepsilon$

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Consider,

$d(p,o) \leq d(p,x) + d(x,o)$ $\because$ by triangular inequality

$\Rightarrow d(p,o) \leq \varepsilon + d(x,o)$

$\Rightarrow d(x,o) > d(p,o) - \varepsilon$

But $\varepsilon = d(p,o) - 1$

$\Rightarrow d(x,o) > d(p,o) - (d(p,o) - 1) = 1$

$\Rightarrow d(x,o) > 1$

$\Rightarrow x \in F^c$.

$\Rightarrow B(p,\varepsilon) \subseteq F^c$.

$\Rightarrow F^c$ is open

Hence F is closed.

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(3) Let (X, d) be any metric space.
Let $A \subseteq X$ be a finite subset of X.
Then A is closed.

Solution :-
Let $A = \{x_1, x_2, x_3, \ldots, x_n\}$
To show that A is closed we will prove that A^c is open.
Let $p \in A^c$.
Let $\varepsilon = \min\{d(p, x_i) \mid 1 \leq i \leq n\}$
Clearly $\varepsilon > 0$

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Choose $\delta < \varepsilon$

We will prove that

$$B(p, \delta) \subseteq A^c$$

Let $y \in B(p, \delta)$ $\Rightarrow d(y, p) < \delta < \varepsilon$

$\Rightarrow d(y, p) < \varepsilon$

$\Rightarrow y \notin A$

$\Rightarrow y \in A^c$

$\Rightarrow B(p, \delta) \subseteq A^c$

$\Rightarrow A^c$ is open

$\Rightarrow A$ is closed.