Lecture No 19: Topology of Metric Spaces

Santosh Shivlal Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

ssdhamone@acscollegeonde.ac.in

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Lecture 3 : Unit II : Sequences Complete Metric Space

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Arts,Commerce & Science College,Onde
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Topology of Metric Spaces

Unit II : Sequences and Complete Metric Space

Lecture - 3

Santosh Shivlal Dhamone
Assistant Professor in Mathematics
Arts Commerce and Science College, Onde

Properties of Closed Balls :

Theorem :-

Let (X, d) be a metric space.

(1) ϕ and X are closed sets.

(2) Finite union of closed sets is closed.

(3) Arbitrary intersection of closed sets is closed.

My Inspiration Shri. V.G. Patil Saheb
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Proof :-

(1) To Prove that ϕ & X are closed sets.

That is we have to prove that ϕ^c & X^c are open sets.

$\phi^c = X$

we know that X is open set.

$\Rightarrow \phi$ is closed set.

Similarly, $X^c = \phi$

we know that ϕ is open set.

$\Rightarrow X^c$ is open set.

$\Rightarrow X$ is closed set.

Thus ϕ & X are sets which are both open & closed.

(2) To prove that, finite union of closed sets is closed

Let $F_1, F_2, \ldots, F_n$ be n closed subsets of X.

Let $F = F_1 \cup F_2 \cup F_3 \cup \cdots \cup F_n$

To show that F is closed.

That is we have to show that F^c is open set in X.

$$F^c = (F_1 \cup F_2 \cup F_3 \cup \cdots \cup F_n)^c$$

By De-Morgan's Law,

$$(A \cup B)^c = A^c \cap B^c$$

$$F^c = F_1^c \cap F_2^c \cap F_3^c \cap \cdots \cap F_n^c$$

since $F_1, F_2, \cdots, F_n$ are closed subsets of X.

$\therefore F_1^c, F_2^c, \cdots, F_n^c$ are open sets.

We know that, by theorem in open sets.

Finite intersection of open sets is open.

$\Rightarrow F^c = F_1^c \cap F_2^c \cap \cdots \cap F_n^c$ is open.

$\Rightarrow F^c$ is open

$\Rightarrow F$ is closed set.

Thus finite union of closed sets is closed.

(3) Arbitrary intersection of closed sets is closed.

Let $\mathcal{F} = \{F_\alpha \mid \alpha \in I\}$ be a family of closed set.

I is the indexing set which may be finite or infinite.

Let $F = \bigcap_{\alpha \in I} F_\alpha$

To show that F is closed,

we consider

$$F^c = \left(\bigcap_{\alpha \in I} F_\alpha\right)^c$$

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By De-Morgan's Law

$$(A \cap B)^c = A^c \cup B^c$$

$$F^c = \bigcup_{\alpha \in I} F_\alpha^c$$

F_α is closed in X

$\Rightarrow F_\alpha^c$ is open in X for $\alpha \in I$.

We know that, Arbitrary union of open sets is open.

$\Rightarrow \bigcup_{\alpha \in I} F_\alpha^c$ is open.

$\Rightarrow$ F^c is open.

$\Rightarrow$ F is closed.

Thus arbitrary intersection of closed sets is closed.

Remarks :-

(1) Arbitrary union of closed sets need not be closed.

Let $X = \mathbb{R}$, d = usual metric or standard metric.

Let $F_n = [0, 1-\frac{1}{n}]$ for $n \in \mathbb{N}$.

F_n is closed set as its complement

$$F_n^c = (-\infty, 0) \cup (1-\tfrac{1}{n}, \infty)$$

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F_n^c is a union of open intervals & hence is open.

Consider,
$$F = \bigcup_{n \in \mathbb{N}} [0, 1-\tfrac{1}{n}]$$
$$= [0, 1]$$
which is not closed.

(2) There are sets in a metric space which are neither open nor closed.

For example :-

$F = [0, 1]$ is not closed as its complement $F^c = (-\infty, 0) \cup [1, \infty)$ does not contain any open ball centered at 1 as $B(1, \varepsilon) = (1-\varepsilon, 1+\varepsilon)$ contains point < 1 which does not belong to F^c.

Further F is not open as F does not contain any open ball centered at 0 as $B(0, \varepsilon) = (-\varepsilon, \varepsilon)$ contain point < 0 which does not belong to F. Thus F is neither open nor closed.

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Theorem :- In a metric space, every closed ball is a closed set.

Proof :- Let (X, d) be a metric space.
Let $p \in X$ be arbitrary point &
let $\varepsilon > 0$ be any positive real number.
Let $F = B[p, \varepsilon]$ = closed ball centered at p & radius ε

$$\therefore F = B[p, \varepsilon] = \{ x \in X \mid d(x, p) \leq \varepsilon \}$$

We will prove that F^c is open

$$F^c = \{ x \in X \mid d(x, p) > \varepsilon \}$$

Let $x \in F^c$ be any point,

$d(x, p) > \varepsilon$

$\Rightarrow d(x, p) - \varepsilon > 0$

Take $\delta = d(x, p) - \varepsilon$

We will prove that $B(x, \delta) \subseteq F^c$

Let $y \in B(x, \delta)$

$\Rightarrow d(x, y) < \delta$

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By triangular property of metric space X

$$d(x,p) \le d(x,y) + d(y,p)$$
$$< s + d(y,p)$$

$\Rightarrow d(y,p) > d(x,p) - s$

But $s = d(x,p) - \varepsilon$ i.e. $\varepsilon = d(x,p) -$

$\therefore d(y,p) > \varepsilon$

$\Rightarrow y \in F^c$

$\Rightarrow B(x,s) \subseteq F^c$.

$\Rightarrow F^c$ is open. Hence Every closed ball is a closed set.