Lecture No 20: Topology of Metric Spaces

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Contents

Lecture 4 : Unit II : Sequences Complete Metric Space

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Topology of Metric Spaces

Unit II : Sequences and Complete Metric Space

Lecture - 4

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My Inspiration
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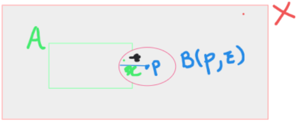
Limit Point and Derived Sets :-

Definition of Limit Point :-

Let (X, d) be a metric space. Let $A \subseteq X$. A point $p \in X$ is said to be limit point of A if every open ball centered at p contains a point of A other than p.



In other words, for every $\varepsilon > 0$, there exists a point $x \neq p$ such that

$$x \in B(p, \varepsilon) \cap A$$

That is, for all $\varepsilon > 0$,

$$[B(p, \varepsilon) \setminus \{p\}] \cap A \neq \emptyset$$

Defination of Derived Sets :-

The set of all limit points of A is called derived set of A and it is denoted by $D(A)$.

Examples :-

Find the limit points of the following subset of $\mathbb{R}$ with the usual metric and also find their derived sets :

1. $\mathbb{N}$
2. $\mathbb{Z}$
3. $\mathbb{Q}$
4. $\{ \frac{1}{n} \mid n \in \mathbb{N} \}$
5. $\{ \frac{1}{3^n} \mid n \in \mathbb{N} \}$
6. $\{ m + \frac{1}{4^n} \mid m, n \in \mathbb{N} \}$

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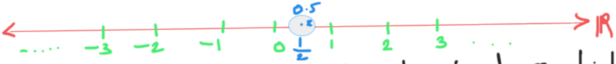
Solutions :-

1. $\mathbb{N}$ = Set of natural numbers. $\mathbb{N} \subseteq \mathbb{R}$.

Suppose $x \in \mathbb{R}$ be any real number.

We will prove that x can not be a limit point of $\mathbb{N}$.



We will produce an open ball centered at x which does not contain a natural number other than possibly x.

For $x \in \mathbb{R}$

There exist $m \in \mathbb{N}$ such that $m < x < m+1$

Let $\varepsilon = \min\{x-m,\ m+1-x\}$

Suppose $x \in \mathbb{N}$, then the interval $(x-\frac{1}{2}, x+\frac{1}{2})$ contains only natural number x & hence

$$\left(B(x,\tfrac{1}{2}) \setminus \{x\}\right) \cap \mathbb{N} = \emptyset$$

$\Rightarrow$ x is not a limit point.

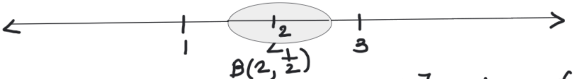
Suppose $x \notin N$, then the interval $(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})$ does not contain any natural number

$\Rightarrow \left(B(x, \frac{\varepsilon}{2}) \setminus \{x\} \right) \cap N = \emptyset$

$\Rightarrow$ No real number is a limit point of N.

$\Rightarrow D(N) = \emptyset$

$\varepsilon = \frac{1}{2}$



$\left[B(2, \frac{1}{2}) \setminus \{2\} \right] \cap N = \emptyset$

Set of natural number N has no limit point.

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3. $\mathbb{Q}$ - Set of Rationals , $\mathbb{Q} \subseteq \mathbb{R}$

Let $x \in \mathbb{R}$ be any real number.
Let $\varepsilon > 0$ be any real number.
Consider any open ball centered at x with radius ε

$\Rightarrow B(x, \varepsilon) = (x-\varepsilon, x+\varepsilon)$ $\because$ By def^n of op ball.

This intervals contains infinitely many rational numbers.

$(B(x, \varepsilon) \setminus \{x\}) \cap \mathbb{Q} \neq \emptyset$

$\Rightarrow x$ is limit point for any $x \in \mathbb{R}$

$\Rightarrow D(\mathbb{Q}) = \mathbb{R}$. $\Rightarrow D(\mathbb{Q}) = \mathbb{R}$.

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2. $\mathbb{Z}$ = Set of integers, $\mathbb{Z} \subseteq \mathbb{R}$.

$B(-1, \frac{1}{2}) \longrightarrow$ not contain any integers except -1.

$$\therefore \left[B(x, \tfrac{\varepsilon}{2}) \setminus \{x\}\right] \cap \mathbb{Z} = \phi$$

$$D(\mathbb{Z}) = \phi$$

It is easy to prove on a similar lines that $\mathbb{Z}$ does not have a limit point & $D(\mathbb{Z}) = \phi$.