QP Code : 24929

($2\frac{1}{2}$ Hours) [Total Marks: 75

N.B.: (1) **All questions are compulsory.**
(2) Figures to the right indicate marks for respective subquestions.

1. (a) Attempt any **one** question:

(i) Let V be a finite dimensional real vector space and W be a subspace of V. Show that $\dim V/W = \dim V - \dim W$. (8)

(ii) Let V be a finite dimensional inner product space over $\mathbb{R}$. If $f : V \to V$ is a map such that (1) $f(0) = 0$ (2) $\|f(x) - f(y)\| = \|x - y\|$, $\forall\, x, y \in V$, then show that f is orthogonal linear transformation. (8)

(b) Attempt any **three** questions:

(i) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(X) = AX$ (X being a column vector in $\mathbb{R}^3$). Find $\ker T$, a basis of $\ker T$ and $\mathbb{R}^3/\ker T$. Also find $\operatorname{Im} T$. (4)

(ii) Let V be an n-dimensional inner product space over $\mathbb{R}$ and let W be a subspace of V such that $\dim W = n - 1$. Let u be a unit vector orthogonal to W. Show that $T : V \to V$ defined by $Tv = v - 2\langle v, u\rangle u$ is an orthogonal linear transformation such that $T(w) = w$, $\forall w \in W$ and $T(u) = -u$. (4)

(iii) Let A be 3×3 orthogonal matrix such that $\det A = 1$. Show that 1 is eigenvalue of A. (4)

(iv) Using Cayley Hamilton Theorem, find A^{100} where $A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$. (4)

2. (a) Attempt any **one** questions:

(i) Let A be an $n \times n$ real matrix. Prove that the following statements are equivalent.

(1) A is diagonalizable.

(2) $\mathbb{R}^n$ has a basis of eigenvectors of A. (8)

(ii) Let A be an $n \times n$ real symmetric matrix. Show that the characteristic roots of A are real. (8)

(b) Attempt any **three** questions:

(i) Let A be an $n \times n$ real matrix. If every non-zero $X \in \mathbb{R}^n$ is an eigenvector of A, then show that $A = \lambda I_n$ for some $\lambda \in \mathbb{R}$. (4)

(ii) Let A be an $n \times n$ matrix which has eigenvalues 1 and -1. If A is diagonalizable then show that $A = A^{-1}$. (4)

(iii) If A and B are $n \times n$ real orthogonally diagonalizable matrices such that $AB = BA$, then show that AB is also orthogonally diagonalizable. (4)

(iv) Show that every quadratic form $Q(x_1, x_2, \cdots, x_n)$ over $\mathbb{R}$ can be reduced to a standard form $\sum_{i=1}^{n} \lambda_i y_i^2$ by an orthogonal change of variables $X = PY$, $X = (x_1, x_2 \cdots, x_n)^t$ and $Y = (y_1, y_2, \cdots, y_n)^t$. (4)

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3. (a) Attempt any **one** questions:

(i) Let G be a cyclic group of order n generated by $'a'$. Show that G has a unique subgroup of order d for each divisor d of n. (8)

(ii) Let G, G' be groups and $f : G \to G'$ be a homomorphism of groups. Show that (8)

(1) $f(e) = e'$ where e, e' are identity elements of G, G' respectively.

(2) $f(a^{-1}) = (f(a))^{-1}$ for each $a \in G$

(3) $f(a^n) = (f(a))^n$ for each $a \in G$ and each $n \in \mathbb{Z}$.

Further if f is onto, then show that

(1) G is abelian implies G' is abelian

(2) G is cyclic and $G = \langle a \rangle$ implies G' is cyclic and $G' = \langle f(a) \rangle$.

(b) Attempt any **three** questions:

(i) Let H, K be subgroups of a group G. Show that HK is a subgroup of G if and only if $HK = KH$. (4)

(ii) Let H be finite subset of group G such that $ab \in H$ whenever a and $b \in H$. Show that H is subgroup of G. (4)

(iii) Exhibit an element of order 20 in S_9, the symmetric group on 9 symbols. (4)

(iv) Let G, G' be groups, $f, g : G \to G'$ be group homomorphisms and

$$H = \{x \in G : f(x) = g(x)\}.$$

Prove or disprove: H is a subgroup of G. (4)

4. Attempt any **three** questions:

(a) Let $V = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ be vector space of all 2×2 upper triangular matrices and $W = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} : a, c \in \mathbb{R} \right\}$ be its subspace. Let $U = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{R} \right\}$. Show that $V/W \cong U$. (5)

(b) Let V be an n-dimensional inner product space. Suppose $B = \{e_i\}_{i=1}^{n}$ and $B' = \{f_i\}_{i=1}^{n}$ are orthonormal bases of V. If $T : V \to V$ is a linear transformation such that $T(e_i) = f_i$ for $i = 1$ to n, then show that T is orthogonal transformation. (5)

(c) Let $u = (a, b, c) \in \mathbb{R}^3$. Find eigenvalues and eigenvectors of $u^t u$ and show that $u^t u$ is diagonalizable. (5)

(d) Let A be any $n \times n$ diagonalizable matrix. Show that For any positive integer k, A^k is also diagonalizable. (5)

(e) If G is a finite abelian group having 2 distinct elements of order 2, show that $4 \mid o(G)$. (5)

(f) Show that the map $f : GL_n(R) \to GL_n(R)$ defined by $f(A) = (A^t)^{-1}$ is a group homomorphism. Is it an automorphism? Justify your answer. (5)
