QP Code : 14570

$(2\frac{1}{2})$ **Hours** **Total Marks : 75**

N.B.: (1) All questions are compulsory
(2) Figures to the right indicate marks for respective subquestions.

Q1a) Attempt any **one** question. (8)

i) Let S be an open subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}^m$ with $f(x) = (f_1(x), f_2(x), \cdots \cdots f_m(x))$. Let $a \in S$. Then show that f is continuous at a iff each f_k is continuous at $a, 1 \leq k \leq m$.

ii) Let S be an open subset of $\mathbb{R}^n$ and $f, g: S \to \mathbb{R}^m$ and let $a \in S$. If $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$ then using $\epsilon - \delta$ definition, prove that (8)

1) $\lim_{x\to a} ||f(x)|| = ||b||$
2) $\lim_{x\to a}(f(x) \cdot g(x)) = b \cdot c$

b) Attempt any **three** questions

i) State and prove the mean value theorem for scalar fields. (4)

ii) Let S be an open subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}$ and let $a \in S$. Define the partial derivative $D_k f$ of f at a. Find the partial derivatives of f at (0,0) for (4)

$$f(x,y) = \begin{cases} \frac{x+y}{x-y}, & if\ x \neq y \\ 0, & if\ x = y \end{cases}$$

iii) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exists. (4)

iv) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \sqrt{|xy|}$. Show that f is continuous at (0,0). (4)

Q2 a) Attempt any **one** question.

i) Let S be an open subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}$ be differentiable at $a \in S$ with total derivative $Df(a)$. Show that $f'(a; y)$ exists for all $y \in \mathbb{R}^n$ and $f'(a;y) = Df(a)$ and $f'(a;y) = \sum_{k=1}^{n} D_k f(a) y_k = \nabla f(a) \cdot y$ for all $y \in \mathbb{R}^n$. (8)

ii) Let S be an open subset of $\mathbb{R}^2$ and $f: S \to \mathbb{R}$ be such that $D_1 f, D_2 f, D_{12} f, D_{21} f$ exists on S. If $(a,b) \in S$ and $D_{12} f, D_{21} f$ are continuous on S, then show that $D_{12} f(a,b) = D_{21} f(a,b)$. (8)

b) Attempt any **three** questions

i) Let $z = f(x,y)$ is a differentiable scalar field and $r(t) = (x(t), y(t))$ is a differentiable function of t on an interval I, then show that $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$. (4)

ii) Let S be an open subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}$ be differentiable on S. If $\nabla f(x) = 0, \forall\, x \in B(a,r)$ for $a \in S$, then show that f is constant on $B(a,r)$. (4)

iii) Determine whether f is differentiable at (0,0) where (4)

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & , if\ (x,y) \neq (0,0) \\ 0 & , if\ (x,y) = (0,0) \end{cases}$$

iv) Given $z = u(x,y)e^{ax+by}$ and $\frac{\partial^2 u}{\partial x \partial y} = 0$. Find the values of the constants a, b such that $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$. (4)

Q3. a) Attempt any **one** question.

i) For the surface r(u,v) described by the vector equation r(u,v) = $X(u,v)\bar{\imath}+Y(u,v)\bar{\jmath}+Z(u,v)\bar{k}$, (u,v) $\in$ T ,where X,Y,Z are differentiable on T in u –v plane. If C is a smooth curve lying on the surface then prove that $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is normal to C at each point. (8)

ii) State and prove Divergence Theorem for a simple solid region V bounded by an orientable closed surface S which can be projected on XY,YZ and XZ planes. (8)

b) Attempt any **three** questions

i) Find the equation of the tangent plane to the given parametric surface r(u,v) =(x(u,v),y(u,v),z(u,v)) for x = u+v, y = u-v, z= u^2-v^2 at (1,0,0). (4)

ii) Evaluate the surface integral $\iint_S z\,ds$, where S is the surface of the cylinder $y^2 + z^2$=1 between the planes x=1 and x=2. (4)

iii) Use stokes' Theorem to evaluate $\int_C \bar{F}.d\bar{r}$, where $\bar{F}$=y $\bar{\imath}$ +xz $\bar{\jmath}$ +$z^2\bar{k}$ and C is the boundary of the surface of the plane z=x+4 cut by the cylinder $x^2 + y^2$ =4. (4)

iv) Assuming S and V satisfy the conditions of the Divergence Theorem and scalar field f and g have continuous partial derivatives ,with usual notation ,prove that (4)

1) $\iint_S (f\nabla g).\hat{n}ds = \iiint_V (f\nabla^2 g + \nabla f.\nabla g)dV$

2) $\iint_S (f\nabla g - g\nabla f).\hat{n}ds = \iiint_V (f\nabla^2 g - g\nabla^2 f)dV$

Q4. Attempt any **three** questions

a) Let f: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that f is continuous at each $a \in \mathbb{R}^n$. (5)

b) Show that the sum of x, y, z intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is constant. (5)

c) If f: $\mathbb{R}^3 \to \mathbb{R}^2$is differentiable at (0,0) and $f(0,0,0) = (1,2)$, $Df(0,0,0) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. If g: $\mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(x, y) = (x + 2y + 1\,,\ 3xy)$, then find $D(g^\circ f)(0,0,0)$. (5)

d) Use chain rule and find $\frac{\partial w}{\partial s}, \frac{\partial w}{\partial t}$ at $s = 0, t = 1$ where $w = xy + yz + zx,\ x(s,t) = s\,t, y(s,t) = e^{st}, z(s,t) = t^2$. (5)

e) Use divergence theorem to evaluate $\iint_S \bar{F}.\hat{n}\, ds$, where $\bar{F}=x^2\bar{\imath} +y^2\bar{\jmath} +z^2\bar{k}$ and S the surface of solid bounded by the paraboloid $z=x^2 + y^2$ and the plane $z=1$. (5)

f) Evaluate the surface integral $\iint_S \bar{F}.\hat{n}ds$, where $\bar{F}=y\,\bar{\imath} +x\bar{\jmath} +z\,\bar{k}$ and S is the surface of the parabolic cylinder $y=x^2$ cut by the planes y=1, z=0 and z=2. (5)
