(3 Hours) [Total Marks: 100

N.B.: (1) All questions are **compulsory**.
(2) In **each** question attempt any **two parts** from is a), b), c).
(3) Figures to the right indicate marks for respective subquestions.

Q 1. (a) (i) When is a metric space (X, d) said to be complete? Prove that a finite metric space is complete. (5)

(ii) Let (X, d) and (Y, d_1) be metric spaces and $f : X \to Y$ be a continuous map. Show that $f(\overline{A}) \subset \overline{f(A)}$, for each subset A of X. (5)

(b) (i) Let (X, d) be a metric space. Define a dense subset in (X, d). If A, B are dense subsets in X and B is an open set such that $A \cap B \neq \emptyset$, then show that $A \cap B$ is dense. (5)

(ii) Assuming Weierstrass approximation theorem for $C[0, 1]$, prove it for $C[a, b]$. (5)

(c) (i) If $K_1, \ldots, K_n$ are compact subsets of a metric space (X, d) then show that $\bigcup_{i=1}^{n} K_i$ is compact. (5)

(ii) Show that $A = \{(x, y) \in \mathbb{R}^2 \mid 1 < 2x + 3y < 6\}$ is a path connected subset of $\mathbb{R}^2$ (distance being Euclidean). (5)

Q 2. (a) State and prove Cantor's Intersection Theorem. (10)

(b) (i) Let (X, d) be a metric space. Let A be a subset of X. Show that p is a limit point of A if and only if there is a sequence (x_n) of distinct points in A converging to p. (6)

(ii) Let d_1 and d_2 be metrics on X such that there exists $k_1, k_2 > 0$ satisfying $k_1 d_1(x, y) \leq d_2(x, y) \leq k_2 d_1(x, y)$, for all $x, y \in X$. Prove that if (X, d_1) is complete then (X, d_2) is complete. (4)

(c) (i) Let (X, d_1) and (Y, d_2) be metric spaces. Show that $B(p, r) \times B(q, s)$ is an open set in the metric space $(X \times Y, d)$, where $p \in X$, $q \in Y$ and (6)

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}.$$

(ii) Let $X = C[0, 1]$ show that $|| \ ||_\infty : X \to \mathbb{R}$ defined by

$$||f||_\infty = \sup\{|f(t)| \ : \ t \in [0, 1]\}$$

is a norm on X. (4)

Q 3. (a) Let (X, d) and (Y, d') be metric spaces and $f : X \to Y$ be a function. Prove that the following are equivalent: (10)

(1) f is continuous on X.
(2) for each open subset G of Y, $f^{-1}(G)$ is open in X.
(3) for each closed subset $F$of Y, $f^{-1}(F)$ is a closed in X.

(b) (i) Let (X, d), (Y, d_1) be metric spaces. Prove that a bijection $f : X \to Y$ is open if and only if it is closed. (6)

(ii) Let (X, d) be a metric space and let $a \in X$. Show that $d_a : X \to \mathbb{R}$ defined by $d_a(x) = d(x, a)$, for all $x \in X$, is uniformly continuous on X. (4)

(c) (i) Show that any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ (distance being Euclidean) is uniformly continuous on $\mathbb{R}^n$. (6)

(ii) Give an example of a function f continuous on A such that $f(A) = B$, where (4)

(1) $A = (0, 1)$ and $B = (0, 1]$

(2) $A = [0, 1] \cup [2, 3]$ and $B = \{0, 1\}$.

Q 4. (a) State and prove Fejer's theorem. (10)

(b) (i) Show that the space $(C[a, b], || \; ||_\infty)$ is complete where (6)

$$||f||_\infty = \sup\{|f(x)| \; : \; x \in [a, b]\}.$$

(ii) Let $f \in C[-\pi, \pi]$ and let $f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$. Show that (4)

$$\sigma_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)(a_k \cos kt + b_k \sin kt).$$

(c) (i) State and prove Bessel's inequality. (6)

(ii) Find the Fourier series of the function (4)

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x < 0 \\ \sin x & \text{for } 0 \le x \le \pi \end{cases}$$

Q 5. (a) If K is a compact subset of $\mathbb{R}^n$ (distance being Euclidean), then show that K is closed and bounded. (10)

(b) (i) Let A be a connected subset of a metric space (X, d). Show that $\overline{A}$ is connected. Is A° connected? Justify your answer. (6)

(ii) Show that $A = \{(x, y) \in \mathbb{R}^2 \mid 1 \le x \le 2\}$ is not a compact subset of $\mathbb{R}^2$ (distance being Euclidean). (4)

(c) (i) If $f : (X, d) \to (Y, d')$ is a continuous function, where (X, d) is a connected set, and (Y, d') is the discrete metric space, then show that f is a constant. (6)

(ii) Prove that $A = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}$ is not a connected subset of $\mathbb{R}^2$ (distance being Euclidean). (4)
