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Practical No-1
Linear transformation, Kernel, Rank-Nullity Theorem

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Linear Transformations

Definition Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $f\colon \mathcal{V} \to \mathcal{W}$ be a function from $\mathcal{V}$ to $\mathcal{W}$. (That is, for each vector $\mathbf{v} \in \mathcal{V}$, $f(\mathbf{v})$ denotes exactly one vector of $\mathcal{W}$.) Then f is a **linear transformation** if and only if both of the following are true:

(1) $f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$, for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$

(2) $f(c\mathbf{v}) = cf(\mathbf{v})$, for all $c \in \mathbb{R}$ and all $\mathbf{v} \in \mathcal{V}$.

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Example

Consider the mapping $f\colon \mathcal{M}_{mn} \to \mathcal{M}_{nm}$, given by $f(\mathbf{A}) = \mathbf{A}^T$ for any $m \times n$ matrix $\mathbf{A}$. We will show that f is a linear transformation.

(1) We must show that $f(\mathbf{A}_1 + \mathbf{A}_2) = f(\mathbf{A}_1) + f(\mathbf{A}_2)$, for matrices $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{M}_{mn}$. However, $f(\mathbf{A}_1 + \mathbf{A}_2) = (\mathbf{A}_1 + \mathbf{A}_2)^T = \mathbf{A}_1^T + \mathbf{A}_2^T$ (by part (2) of Theorem) $= f(\mathbf{A}_1) + f(\mathbf{A}_2)$.

(2) We must show that $f(c\mathbf{A}) = cf(\mathbf{A})$, for all $c \in \mathbb{R}$ and for all $\mathbf{A} \in \mathcal{M}_{mn}$. However, $f(c\mathbf{A}) = (c\mathbf{A})^T = c(\mathbf{A}^T)$ (by part (3) of Theorem 1.12) $= cf(\mathbf{A})$.

Hence, f is a linear transformation. ∎

Example

Consider the function $g : \mathcal{P}_n \to \mathcal{P}_{n-1}$ given by $g(\mathbf{p}) = \mathbf{p}'$, the derivative of $\mathbf{p}$. We will show that g is a linear transformation.

(1) We must show that $g(\mathbf{p}_1 + \mathbf{p}_2) = g(\mathbf{p}_1) + g(\mathbf{p}_2)$, for all $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{P}_n$. Now, $g(\mathbf{p}_1 + \mathbf{p}_2) = (\mathbf{p}_1 + \mathbf{p}_2)'$. From calculus we know that the derivative of a sum is the sum of the derivatives, so $(\mathbf{p}_1 + \mathbf{p}_2)' = \mathbf{p}_1' + \mathbf{p}_2' = g(\mathbf{p}_1) + g(\mathbf{p}_2)$.

(2) We must show that $g(c\mathbf{p}) = cg(\mathbf{p})$, for all $c \in \mathbb{R}$ and $\mathbf{p} \in \mathcal{P}_n$. Now, $g(c\mathbf{p}) = (c\mathbf{p})'$. Again, from calculus we know that the derivative of a constant times a function is equal to the constant times the derivative of the function, so $(c\mathbf{p})' = c(\mathbf{p}') = cg(\mathbf{p})$.

Hence, g is a linear transformation. ∎

Example

Let $\mathcal{V}$ be a finite dimensional vector space, and let B be an ordered basis for $\mathcal{V}$. Then every element $\mathbf{v} \in \mathcal{V}$ has its coordinatization $[\mathbf{v}]_B$ with respect to B. Consider the mapping $f : \mathcal{V} \to \mathbb{R}^n$ given by $f(\mathbf{v}) = [\mathbf{v}]_B$. We will show that f is a linear transformation.

Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. By Theorem 4.20, $[\mathbf{v}_1 + \mathbf{v}_2]_B = [\mathbf{v}_1]_B + [\mathbf{v}_2]_B$. Hence,

$$f(\mathbf{v}_1 + \mathbf{v}_2) = [\mathbf{v}_1 + \mathbf{v}_2]_B = [\mathbf{v}_1]_B + [\mathbf{v}_2]_B = f(\mathbf{v}_1) + f(\mathbf{v}_2).$$

Next, let $c \in \mathbb{R}$ and $\mathbf{v} \in \mathcal{V}$. Again by Theorem 4.20, $[c\mathbf{v}]_B = c[\mathbf{v}]_B$. Hence,

$$f(c\mathbf{v}) = [c\mathbf{v}]_B = c[\mathbf{v}]_B = cf(\mathbf{v}).$$

Thus, f is a linear transformation from $\mathcal{V}$ to $\mathbb{R}^n$. ■

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Linear Operators and Some Geometric Examples

An important type of linear transformation is one that maps a vector space to itself.

> **Definition** Let $\mathcal{V}$ be a vector space. A **linear operator** on $\mathcal{V}$ is a linear transformation whose domain and codomain are both $\mathcal{V}$.

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Example

Reflections: Consider the mapping $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f([a_1, a_2, a_3]) = [a_1, a_2, -a_3]$. This mapping "reflects" the vector $[a_1, a_2, a_3]$ through the xy-plane, which acts like a "mirror" (see

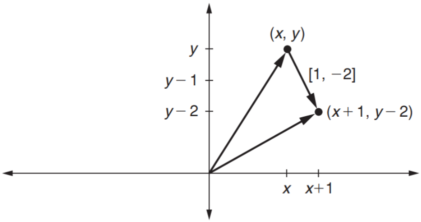

FIGURE 5.1

A translation in $\mathbb{R}^2$

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Now, since

$$\begin{aligned} f([a_1,a_2,a_3]+[b_1,b_2,b_3]) &= f([a_1+b_1,a_2+b_2,a_3+b_3]) \\ &= [a_1+b_1,a_2+b_2,-(a_3+b_3)] \\ &= [a_1,a_2,-a_3]+[b_1,b_2,-b_3] \\ &= f([a_1,a_2,a_3])+f([b_1,b_2,b_3]), \quad \text{and} \end{aligned}$$

$$f(c[a_1,a_2,a_3]) = f([ca_1,ca_2,ca_3]) = [ca_1,ca_2,-ca_3] = c[a_1,a_2,-a_3] = cf([a_1,a_2,a_3]),$$

we see that f is a linear operator. Similarly, reflection through the xz-plane or the yz-plane is also a linear operator on $\mathbb{R}^3$

$\blacksquare$

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Kernel and Image of a Linear Transformation

This section is devoted to two important subspaces associated with a linear transformation $T : V \to W$.

Definition **Kernel and Image of a Linear Transformation**

The ***kernel*** *of* T *(denoted* $\ker T$*) and the* ***image*** *of* T *(denoted* $\operatorname{im} T$ *or* $T(V)$*) are defined by*

$$\ker T = \{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) = \mathbf{0}\}$$
$$\operatorname{im} T = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\} = T(V)$$

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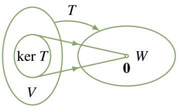

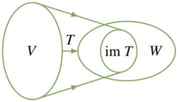

The kernel of T is often called the **nullspace** of T because it consists of all vectors $\mathbf{v}$ in V satisfying the *condition* that $T(\mathbf{v}) = \mathbf{0}$. The image of T is often called the **range** of T and consists of all vectors $\mathbf{w}$ in W of the *form* $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V. These subspaces are depicted in the diagrams.

Example

Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by the $m \times n$ matrix A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. Then

$$\ker T_A = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} = \text{null } A \quad \text{and}$$
$$\text{im } T_A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} = \text{im } A$$

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One-to-One and Onto Transformations

Definition **One-to-one and Onto Linear Transformations**

Let $T : V \to W$ be a linear transformation.

1. T is said to be **onto** if $\operatorname{im} T = W$.
2. T is said to be **one-to-one** if $T(\mathbf{v}) = T(\mathbf{v}_1)$ implies $\mathbf{v} = \mathbf{v}_1$.

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Theorem

If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Proof. If T is one-to-one, let $\mathbf{v}$ be any vector in $\ker T$. Then $T(\mathbf{v}) = \mathbf{0}$, so $T(\mathbf{v}) = T(\mathbf{0})$. Hence $\mathbf{v} = \mathbf{0}$ because T is one-to-one. Hence $\ker T = \{\mathbf{0}\}$.

Conversely, assume that $\ker T = \{\mathbf{0}\}$ and let $T(\mathbf{v}) = T(\mathbf{v}_1)$ with $\mathbf{v}$ and $\mathbf{v}_1$ in V. Then $T(\mathbf{v} - \mathbf{v}_1) = T(\mathbf{v}) - T(\mathbf{v}_1) = \mathbf{0}$, so $\mathbf{v} - \mathbf{v}_1$ lies in $\ker T = \{\mathbf{0}\}$. This means that $\mathbf{v} - \mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v} = \mathbf{v}_1$, proving that T is one-to-one. □

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Example

The identity transformation $1_V : V \to V$ is both one-to-one and onto for any vector space V.

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Example

Consider the linear transformations

$$S : \mathbb{R}^3 \to \mathbb{R}^2 \quad \text{given by } S(x,\ y,\ z) = (x+y,\ x-y)$$
$$T : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{given by } T(x,\ y) = (x+y,\ x-y,\ x)$$

Show that T is one-to-one but not onto, whereas S is onto but not one-to-one.

Solution. The verification that they are linear is omitted. T is one-to-one because

$$\ker T = \{(x,\ y) \mid x+y = x-y = x = 0\} = \{(0,\ 0)\}$$

However, it is not onto. For example $(0,\ 0,\ 1)$ does not lie in $\operatorname{im} T$ because if $(0,\ 0,\ 1) = (x+y,\ x-y,\ x)$ for some x and y, then $x+y = 0 = x-y$ and $x = 1$, an impossibility. Turning to S, it is not one-to-one by Theorem 7.2.2 because $(0,\ 0,\ 1)$ lies in $\ker S$. But every element $(s,\ t)$ in $\mathbb{R}^2$ lies in $\operatorname{im} S$ because $(s,\ t) = (x+y,\ x-y) = S(x,\ y,\ z)$ for some x, y, and z (in fact, $x = \frac{1}{2}(s+t)$, $y = \frac{1}{2}(s-t)$, and $z = 0$). Hence S is onto.

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Theorem

Let A be an $m \times n$ matrix, and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$.

1. *T_A is onto if and only if $\operatorname{rank} A = m$.*
2. *T_A is one-to-one if and only if $\operatorname{rank} A = n$.*

Proof.

1. We have that $\operatorname{im} T_A$ is the column space of A , so T_A is onto if and only if the column space of A is $\mathbb{R}^m$. Because the rank of A is the dimension of the column space, this holds if and only if $\operatorname{rank} A = m$.
2. $\ker T_A = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$, so (using Theorem) T_A is one-to-one if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. This is equivalent to $\operatorname{rank} A = n$ by Theorem $\square$

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Theorem : Dimension Theorem

Let $T : V \to W$ *be any linear transformation and assume that* $\ker T$ *and* $\operatorname{im} T$ *are both finite dimensional. Then* V *is also finite dimensional and*

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

In other words, $\dim V = \operatorname{nullity}(T) + \operatorname{rank}(T)$.

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Example

Let A be an $m \times n$ matrix of rank r. Show that the space null A of all solutions of the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous equations in n variables has dimension $n - r$.

Solution. The space in question is just $\ker T_A$, where $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. But $\dim(\operatorname{im} T_A) = \operatorname{rank} T_A = \operatorname{rank} A = r$ by Example 7.2.2, so $\dim(\ker T_A) = n - r$ by the dimension theorem.

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Example. *The image of*

$$f(x) = e^x$$

consists of all positive numbers.

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Example. *Describe the image of the linear transformation T from R^2 to R^2 given by the matrix*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Solution

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= (x_1 + 3x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example. *Describe the image of the linear transformation* T *from* R^2 *to* R^3 *given by the matrix*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$