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Practical No-2
Linear Isomorphism, Matrix associated with Linear transformations.

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THE MATRIX OF A LINEAR TRANSFORMATION

In this section we will show that a linear transformation between finite-dimensional vector spaces is uniquely determined if we know its action on an ordered basis for the domain. We will also show that every linear transformation between finite-dimensional vector spaces has a unique matrix $\boldsymbol{A}_{BC}$ with respect to the ordered bases B and C chosen for the domain and codomain, respectively.

A Linear Transformation is Determined by its Action on a Basis

One of the most useful properties of linear transformations is that, if we know how a linear map $T : V \to W$ acts on a basis of V, then we know how it acts on the whole of V.

THEOREM Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be an ordered basis for a vector space V. Let W be a vector space, and let $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ be any n (not necessarily distinct) vectors in W. Then there is one and only one linear transformation $T : V \to W$ satisfying $T(\boldsymbol{v}_1) = \boldsymbol{w}_1$, $T(\boldsymbol{v}_2) = \boldsymbol{w}_2$, ..., $T(\boldsymbol{v}_n) = \boldsymbol{w}_n$. In other words, a linear transformation is determined by its action on a basis.

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EXAMPLE Suppose $L : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with

$$L([1, -1, 0]) = [2, 1],\ L([0, 1, -1]) = [-1, 3] \text{ and } L([0, 1, 0]) = [0, 1].$$

Find $L([-1, 1, 2])$. Also, give a formula for $L([x, y, z])$, for any $[x, y, z] \in \mathbb{R}^3$.

[*Delhi Univ. GE-2, 2017*]

SOLUTION To find $L([-1, 1, 2])$, we need to express the vector $\boldsymbol{v} = [-1, 1, 2]$ as a linear combination of vectors $\boldsymbol{v}_1 = [1, -1, 0]$, $\boldsymbol{v}_2 = [0, 1, -1]$ and $\boldsymbol{v}_3 = [0, 1, 0]$. That is, we need to find constants a_1, a_2 and a_3 such that

$$\boldsymbol{v} = a_1 \boldsymbol{v}_1 + a_2 \boldsymbol{v}_2 + a_3 \boldsymbol{v}_3,$$

which leads to the linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 2 \end{array}\right]$$

We transform this matrix to reduced row echelon form :

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

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This gives $a_1 = -1$, $a_2 = -2$, and $a_3 = 2$. So,

$$\begin{aligned} \boldsymbol{v} &= -\boldsymbol{v}_1 - 2\boldsymbol{v}_2 + 2\boldsymbol{v}_3 \\ \Rightarrow \quad L(\boldsymbol{v}) &= L(-\boldsymbol{v}_1 - 2\boldsymbol{v}_2 + 2\boldsymbol{v}_3) \\ &= L(-\boldsymbol{v}_1) - 2L(\boldsymbol{v}_2) + 2L(\boldsymbol{v}_3) \\ &= -[2,\ 1] - 2[-1,\ 3] + 2[0,\ 1] = [0,\ -5] \end{aligned}$$

i.e., $L([-1,\ 1,\ 2]) = [0,\ -5]$

To find $L([x,\ y,\ z])$ for any $[x,\ y,\ z] \in \mathbb{R}^3$, we row reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ -1 & 1 & 1 & y \\ 0 & -1 & 0 & z \end{array}\right] \text{ to obtain } \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & -z \\ 0 & 0 & 1 & x+y+z \end{array}\right]$$

Thus,

$$\begin{aligned} [x,\ y,\ z] &= x\,\boldsymbol{v}_1 - z\,\boldsymbol{v}_2 + (x+y+z)\,\boldsymbol{v}_3 \\ \Rightarrow \quad L([x,\ y,\ z]) &= L(x\,\boldsymbol{v}_1 - z\,\boldsymbol{v}_2 + (x+y+z)\,\boldsymbol{v}_3) \\ &= x\,L(\boldsymbol{v}_1) - z\,L(\boldsymbol{v}_2) + (x+y+z)L(\boldsymbol{v}_3) \\ &= x[2,\ 1] - z[-1,\ 3] + (x+y+z)\,[0,\ 1] \\ &= [2x+z,\ 2x+y-2z]. \end{aligned}$$

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EXAMPLE Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator and $L([1, 1]) = [1, -3]$ and $L([-2, 3]) = [-4, 2]$. Express $L([1, 0])$ and $L([0, 1])$ as linear combinations of the vectors $[1, 0]$ and $[0, 1]$. ***[Delhi Univ. GE-2, 2019]***

SOLUTION To find $L([1, 0])$ and $L([0, 1])$, we first express the vectors $\mathbf{v}_1 = [1, 0]$ and $\mathbf{v}_2 = [0, 1]$ as linear combinations of vectors $\mathbf{w}_1 = [1, 1]$ and $\mathbf{w}_2 = [-2, 3]$. To do this, we row reduce the augmented matrix

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$$[w_1\ w_2 \mid v_1\ v_2] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

Thus, we row reduce

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 5 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{5}R_2} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -1/5 & 1/5 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 3/5 & 2/5 \\ 0 & 1 & -1/5 & 1/5 \end{array}\right]$$

$$\Rightarrow \quad v_1 = \frac{3}{5}w_1 - \frac{1}{5}w_2 \quad \text{and} \quad v_2 = \frac{2}{5}w_1 + \frac{1}{5}w_2$$

$$\Rightarrow \quad \boldsymbol{v}_1 = \frac{3}{5}\boldsymbol{w}_1 - \frac{1}{5}\boldsymbol{w}_2 \quad \text{and} \quad \boldsymbol{v}_2 = \frac{2}{5}\boldsymbol{w}_1 + \frac{1}{5}\boldsymbol{w}_2$$

This gives

$$\begin{aligned} L(\boldsymbol{v}_1) &= \frac{3}{5}L(\boldsymbol{w}_1) - \frac{1}{5}L(\boldsymbol{w}_2) \\ &= \frac{3}{5}L([1,\ 1]) - \frac{1}{5}L([-2,\ 3]) \\ &= \frac{3}{5}[1,\ -3] - \frac{1}{5}[-4,\ 2] = \left[\frac{7}{5},\ \frac{-11}{5}\right] = \frac{7}{5}[1,\ 0] - \frac{11}{5}[0,\ 1] \end{aligned}$$

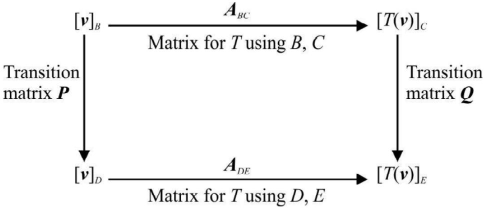
and

$$\begin{aligned} L(\boldsymbol{v}_2) &= \frac{2}{5}L(\boldsymbol{w}_1) + \frac{1}{5}L(\boldsymbol{w}_2) \\ &= \frac{2}{5}L([1,\ 1]) + \frac{1}{5}L([-2,\ 3]) \\ &= \frac{2}{5}[1,\ -3] + \frac{1}{5}[-4,\ 2] = \left[\frac{-2}{5},\ \frac{-4}{5}\right] = -\frac{2}{5}[1,\ 0] - \frac{4}{5}[0,\ 1]. \end{aligned}$$

Finding the New Matrix for a Linear Transformation After a Change of Basis

We now state a theorem (proof omitted) which helps us in computing the matrix for a linear transformation when we change the bases for the domain and codomain.

THEOREM Let V and W be two non-trivial finite-dimensional vector spaces with ordered bases B and C, respectively. Let $T : V \to W$ be a linear transformation with matrix $\mathbf{A}_{BC}$ with respect to bases B and C. Suppose that D and E are other ordered bases for V and W, respectively. Let $\mathbf{P}$ be the transition matrix from B to D, and let $\mathbf{Q}$ be the transition matrix from C to E. Then the matrix $\mathbf{A}_{DE}$ for T with respect to bases D and E is given by $\mathbf{A}_{DE} = \mathbf{Q}\mathbf{A}_{BC}\,\mathbf{P}^{-1}$.



EXAMPLE Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by $T[(a, b, c)] = [-2a + b, -b - c, a + 3c]$.

(*a*) Find the matrix A_{BB} for T with respect to the standard basis $B = \{e_1 = [1, 0, 0], e_2 = [0, 1, 0], e_3 = [0, 0, 1]\}$ for $\mathbb{R}^3$.

(*b*) Use part (*a*) to find the matrix A_{DE} with respect to the standard bases $D = \{[15, -6, 4], [2, 0, 1], [3, -1, 1]\}$ and $E = \{[1, -3, 1], [0, 3, -1], [2, -2, 1]\}$.

SOLUTION (*a*) We have $T(e_1) = [-2, 0, 1]$, $T(e_2) = [1, -1, 0]$, $T(e_3) = [0, -1, 3]\}$. Using each of these vectors as columns yields the matrix A_{BB}:

$$A_{BB} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix}$$

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(*b*) To find A_{DE}, we make use of the following relationship :

$$A_{DE} = QA_{BB}P^{-1}, \qquad \ldots(1)$$

where P is the transition matrix from B to D and Q is the transition matrix from B to E. Since P^{-1} is the transition matrix from D to B and B is the standard basis for $\mathbb{R}^3$, it is given by

$$P^{-1} = \begin{bmatrix} 15 & 2 & 3 \\ -6 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix}$$

To find Q, we first find Q^{-1}, the transition matrix from E to B, which is given by

$$Q^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

It can be easily checked that

$$Q = (Q^{-1})^{-1} = \begin{bmatrix} 1 & -2 & -6 \\ 1 & -1 & -4 \\ 0 & 1 & 3 \end{bmatrix}$$

Hence, using Eq. (1), we obtain

$$A_{DE} = QA_{BB}P^{-1} = \begin{bmatrix} 1 & -2 & -6 \\ 1 & -1 & -4 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 15 & 2 & 3 \\ -6 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -202 & -32 & -43 \\ -146 & -23 & -31 \\ 83 & 14 & 18 \end{bmatrix}.$$

EXAMPLE Let $T : \mathcal{P}_3 \to \mathbb{R}^3$ be the linear transformation given by $T(ax^3 + bx^2 + cx + d) = [c + d, 2b, a - d]$.

(*a*) Find the matrix $\boldsymbol{A}_{BC}$ for T with respect to the standard bases B (for $\mathcal{P}_3$) and C (for $\mathbb{R}^3$).

(*b*) Use part (*a*) to find the matrix $\boldsymbol{A}_{DE}$ for T with respect to the standard bases $D = \{x^3 + x^2, x^2 + x, x + 1, 1\}$ for $\mathcal{P}_3$ and $E = \{[-2, 1, -3], [1, -3, 0], [3, -6, 2]\}$ for $\mathbb{R}^3$.

SOLUTION (*a*) To find the matrix $\boldsymbol{A}_{BC}$ for T with respect to the standard bases $B = \{x^3, x^2, x, 1\}$ for $\mathcal{P}_3$ and $C = \{\boldsymbol{e}_1 = [1, 0, 0], \boldsymbol{e}_2 = [0, 1, 0], \boldsymbol{e}_3 = [0, 0, 1]\}$ for $\mathbb{R}^3$, we first need to find $T(\boldsymbol{v})$ for each $\boldsymbol{v} \in B$. By definition of T, we have

$$T(x^3) = [0, 0, 1], \quad T(x^2) = [0, 2, 0], \quad T(x) = [1, 0, 0] \quad \text{and} \quad T(1) = [1, 0, -1]$$

Since we are using the standard basis C for $\mathbb{R}^3$, the matrix $\boldsymbol{A}_{BC}$ for T is the matrix whose columns are these images. Thus

$$\boldsymbol{A}_{BC} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

(*b*) To find $\boldsymbol{A}_{DE}$, we make use of the following relationship :

$$\boldsymbol{A}_{DE} = \boldsymbol{Q}\boldsymbol{A}_{BC}\boldsymbol{P}^{-1} \qquad \ldots(1)$$

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where $\boldsymbol{P}$ is the transition matrix from B to D and Q is the transition matrix C to E. Since $\boldsymbol{P}$ is the transition matrix from B to D, therefore $\boldsymbol{P}^{-1}$ is the transition matrix from D to B. To compute $\boldsymbol{P}^{-1}$, we need to convert the polynomials in D into vectors in $\mathbb{R}^4$. This is done by converting each polynomial $ax^3 + bx^2 + cx + d$ in D to $[a, b, c, d]$. Thus

$$(x^3 + x^2) \to [1, 1, 0, 0]; \quad (x^2 + x) \to [0, 1, 1, 0]; \quad (x + 1) \to [0, 0, 1, 1]; \quad (1) \to [0, 0, 0, 1]$$

Since B is the standard basis for $\mathbb{R}^3$, the transition matrix ($\boldsymbol{P}^{-1}$) from D to B is obtained by using each of these vectors as columns :

$$\boldsymbol{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

To find $\boldsymbol{Q}$, we first find $\boldsymbol{Q}^{-1}$, the transition matrix from E to C, which is the matrix whose columns are the vectors in E.

$$\boldsymbol{Q}^{-1} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & -6 \\ -3 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \qquad \boldsymbol{Q} = (\boldsymbol{Q}^{-1})^{-1} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & -6 \\ -3 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix}$$

$$\text{Hence,} \quad \boldsymbol{A}_{DE} = \boldsymbol{Q}\boldsymbol{A}_{BC}\boldsymbol{P}^{-1} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix}.$$

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Isomorphisms and Composition

Often two vector spaces can consist of quite different types of vectors but, on closer examination, turn out to be the same underlying space displayed in different symbols. For example, consider the spaces

$$\mathbb{R}^2 = \{(a,\, b) \mid a,\, b \in \mathbb{R}\} \quad \text{and} \quad \mathbf{P}_1 = \{a + bx \mid a,\, b \in \mathbb{R}\}$$

Compare the addition and scalar multiplication in these spaces:

$$\begin{aligned} (a,\, b) + (a_1,\, b_1) &= (a + a_1,\, b + b_1) & \qquad (a + bx) + (a_1 + b_1 x) &= (a + a_1) + (b + b_1)x \\ r(a,\, b) &= (ra,\, rb) & r(a + bx) &= (ra) + (rb)x \end{aligned}$$

Clearly these are the *same* vector space expressed in different notation: if we change each $(a,\, b)$ in $\mathbb{R}^2$ to $a + bx$, then $\mathbb{R}^2$ *becomes* $\mathbf{P}_1$, complete with addition and scalar multiplication. This can be expressed by noting that the map $(a,\, b) \mapsto a + bx$ is a linear transformation $\mathbb{R}^2 \to \mathbf{P}_1$ that is both one-to-one and onto. In this form, we can describe the general situation.

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Definition **Isomorphic Vector Spaces**

A linear transformation $T : V \to W$ is called an ***isomorphism*** *if it is both onto and one-to-one. The vector spaces V and W are said to be* ***isomorphic*** *if there exists an isomorphism $T : V \to W$, and we write $V \cong W$ when this is the case.*

Example

The identity transformation $1_V : V \to V$ is an isomorphism for any vector space V.

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Theorem

If V and W are finite dimensional vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

Proof. It remains to show that if $V \cong W$ then $\dim V = \dim W$. But if $V \cong W$, then there exists an isomorphism $T : V \to W$. Since V is finite dimensional, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of V. Then $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)\}$ is a basis of W by Theorem 7.3.1, so $\dim W = n = \dim V$. □

Corollary

Let U, V, and W denote vector spaces. Then:

1. *$V \cong V$ for every vector space V.*
2. *If $V \cong W$ then $W \cong V$.*
3. *If $U \cong V$ and $V \cong W$, then $U \cong W$.*

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Corollary

If V is a vector space and $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$.

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Example

Let V denote the space of all 2×2 symmetric matrices. Find an isomorphism $T:\mathbf{P}_2 \to V$ such that $T(1) = I$, where I is the 2×2 identity matrix.

Solution. $\{1, x, x^2\}$ is a basis of $\mathbf{P}_2$, and we want a basis of V containing I. The set $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is independent in V, so it is a basis because $\dim V = 3$ (by Example 6.3.11). Hence define $T:\mathbf{P}_2\to V$ by taking $T(1) = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $T(x) = \begin{bmatrix}0&1\\1&0\end{bmatrix}$, $T(x^2) = \begin{bmatrix}0&0\\0&1\end{bmatrix}$, and extending linearly as in Theorem 7.1.3. Then T is an isomorphism by Theorem 7.3.1, and its action is given by

$$T(a+bx+cx^2) = aT(1)+bT(x)+cT(x^2) = \begin{bmatrix}a&b\\b&a+c\end{bmatrix}$$

Example

Define: $S : \mathbf{M}_{22} \to \mathbf{M}_{22}$ and $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ by $S\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ and $T(A) = A^T$ for $A \in \mathbf{M}_{22}$. Describe the action of ST and TS, and show that $ST \neq TS$.

Solution. $ST\begin{bmatrix} a & b \\ c & d \end{bmatrix} = S\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$, whereas

$TS\begin{bmatrix} a & b \\ c & d \end{bmatrix} = T\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$.

It is clear that $TS\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ need not equal $ST\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so $TS \neq ST$.