Practical No-3
Inner product and properties, Projection, Orthogonal complements.

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Definition

A general inner product in a real (complex) vector space $\mathcal{V}$ is a symmetric (Hermitian) bilinear form $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}\ (\mathbb{C})$, i.e.,

1. $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \in \mathbb{R}_{\geq 0}$ with $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$.
2. $\langle \boldsymbol{x}, \alpha\boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ for all scalars α.
3. $\langle \boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle$.
4. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$ (or $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}$ if complex).

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As before, any inner product induces a norm via

$$\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}.$$

One can show (analogous to the Euclidean case) that $\|\cdot\|$ is a norm.

In particular, we have a general Cauchy–Schwarz–Bunyakovsky inequality

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|.$$

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Example

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ (or $\mathbf{x}^* \mathbf{y}$), the standard inner product for $\mathbb{R}^n$ ($\mathbb{C}^n$).
2. For nonsingular matrices A we get the A-inner product on $\mathbb{R}^n$, i.e.,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathrm{A}^T \mathrm{A} \mathbf{y}$$

with

$$\|\mathbf{x}\|_\mathrm{A} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x}^T \mathrm{A}^T \mathrm{A} \mathbf{x}} = \|\mathrm{A}\mathbf{x}\|_2.$$

3. If $\mathcal{V} = \mathbb{R}^{m \times n}$ (or $\mathbb{C}^{m \times n}$) then we get the standard inner product for matrices, i.e.,

$$\langle \mathrm{A}, \mathrm{B} \rangle = \mathrm{trace}(\mathrm{A}^T \mathrm{B}) \quad (\text{or } \mathrm{trace}(\mathrm{A}^* \mathrm{B}))$$

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Parallelogram identity

In any inner product space the so-called parallelogram identity holds, i.e.,

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right). \tag{2}$$

This is true since

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle + \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \\ &= \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle \\ &\quad + \langle \mathbf{x}, \mathbf{x}\rangle - \langle \mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle \\ &= 2\langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{y}, \mathbf{y}\rangle = 2\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right).\end{aligned}$$

Orthogonal Complements and Projections

Recall that two vectors v_1 & v_2 in $\mathbb{R}^n$ are *perpendicular* or *orthogonal* provided that their *dot product* vanishes. That is, $v_1 \perp v_2$ if and only if $v_1 \cdot v_2 = 0$.

Example

1. The vectors $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ & $\begin{pmatrix} 12 \\ 8 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$ are orthogonal while $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ & $\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}$ are not.

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2. We can define an *inner product* on the vector space of all polynomials of degree at most 3 by setting

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)\, g(x)\, dx.$$

(There is nothing special about integrating over [0,1]; This interval was chosen arbitrarily.) Then, for example,

$$\begin{aligned}\langle 2x^2+1, 10x^2+11x-11 \rangle &= \int_0^1 \left(2x^2+1\right)\left(10x^2+11x-11\right) dx \\ &= \int_0^1 \left(20x^4+22x^3-12x^2+11x-11\right) dx \\ &= \left(4x^5+\frac{11}{2}x^4-4x^3+\frac{11}{2}x^2-11x\right)\Big|_0^1 \\ &= 0\end{aligned}$$

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Hence, relative to the inner product $\langle f(x), g(x) \rangle = \int_0^1 f(x)\, g(x)\, dx$ we have that the two polynomials $2x^2 + 1$ & $10x^2 + 11x - 11$ are *orthogonal* in P_3.

So, more generally, we say that $v_1 \perp v_2$ in a vector space V with inner product $\langle u, v \rangle$ provided

that $\langle u, v \rangle = 0$.

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Suppose V is a vector space with inner product $\langle u, v \rangle$. (Think $V = \mathbb{R}^n$ and $\langle u, v \rangle = dot(u, v)$)

1. The subspaces S_1 & S_2 of $\mathbb{R}^n$ are said to be ***orthogonal***, denoted $S_1 \perp S_2$, if $\langle v_1, v_2 \rangle = 0$ for all $v_1 \in S_1$ & $v_2 \in S_2$.

2. Let W be a subspace of V. Then we define $W^\perp$ (read "*W perp*") to be the set of vectors in V given by

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W \}.$$

The set $W^\perp$ is called the ***orthogonal complement*** of W.

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Examples

1. From the above work, if $A = \begin{pmatrix} 1 & 1 & 3 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 0 & 8 & 2 \end{pmatrix}$, then $R_A \perp N_A$.

2. Let A be any $m \times n$ matrix. Now, the null space N_A of A consists of those vectors x with $A\,x = 0_m$. However, $A\,x = 0_m$ if and only if $r_i \cdot x = 0 \quad (i = 1, \ldots, m)$ for each row r_i of the matrix A. Hence, the null space of A is the set of all vectors orthogonal to the rows of A and, hence, the row space of A. (Why?) We conclude that $R_A^{\perp} = N_A$.

The above suggest the following method for finding $W^{\perp}$ given a subspace W of $\mathbb{R}^n$.

1. Find a matrix A having as row vectors a generating set for W.
2. Find the null space of A. This null space is $W^{\perp}$.

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3. Suppose that $S_1 = span\left\{\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}\right\}$ and $S_2 = span\left\{\begin{pmatrix}0\\-1\\1\\1\\0\end{pmatrix}\right\}$. Then S_1 & S_2

are *orthogonal* subspaces of $\mathbb{R}^5$. To verify this observe that

$$\left(a\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}\right)\cdot\left(r\begin{pmatrix}0\\-1\\1\\1\\0\end{pmatrix}\right) = a\,r\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\1\\1\\0\end{pmatrix} + b\,r\begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\1\\1\\0\end{pmatrix}$$

$$= a\,r\,(0) + b\,r\,(0)$$

$$= 0$$

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Thus, $\boldsymbol{S_1} \perp \boldsymbol{S_2}$. Since

$$\left(a\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} -7 \\ -2 \\ 2 \\ 2 \\ 5 \end{pmatrix} = 0$$

and

$$\begin{pmatrix} -7 \\ -2 \\ 2 \\ 2 \\ 5 \end{pmatrix} \notin \boldsymbol{S_2},$$

it follows that $S_1^\perp \neq S_2$. So, what is the set $S_1^\perp$? Let $B = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$.

Then, from part 2 above, $S_1^\perp = N_B$. In fact, a basis for $S_1^\perp = N_B$ can be shown to be

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Finally, we note that the set $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ forms a basis

for $\mathbb{R}^5$. In particular, every element of $\mathbb{R}^5$ can be written as the sum of a vector in S_1 and a vector in $S_1^\perp$.

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4. Let W be the subspace of P_3 (= the vector space of all polynomials of degree at most 3) with basis $\{1, x^3\}$. We take as our inner product on P_3 the function

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)\, g(x)\, dx.$$

Find as basis for $W^\perp$.

Solution

Let $p(x) = ax^3 + bx^2 + cx + d \in W^\perp$. Then

$$\langle p(x), g(x) \rangle = \int_0^1 p(x)\, g(x)\, dx = 0$$

for all $g(x) \in W$. Hence, in particular,

$$\langle p(x), 1 \rangle = \int_0^1 \left(ax^3 + bx^2 + cx + d\right) dx = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0$$

and

$$\langle p(x), x^3 \rangle = \int_0^1 \left(ax^6 + bx^5 + cx^4 + dx^3\right) dx = \frac{a}{7} + \frac{b}{6} + \frac{c}{5} + \frac{d}{4} = 0.$$

Solving the linear system

$$\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0$$

$$\frac{a}{7} + \frac{b}{6} + \frac{c}{5} + \frac{d}{4} = 0$$

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we find that we have pivot variables of $a = \frac{14}{5}c + 14d$ and $b = -\frac{18}{5}c - \frac{27}{2}d$ with free variables of c and d. It follows that

$$p(x) = c\left(\frac{14}{5}x^3 - \frac{18}{5}x^2 + x\right) + d\left(14x^3 - \frac{27}{2}x^2 + 1\right)$$

for some $c, d \in \mathbb{R}$. Hence, the polynomials

$$\frac{14}{5}x^3 - \frac{18}{5}x^2 + x \quad \& \quad 14x^3 - \frac{27}{2}x^2 + 1$$

span $W^{\perp}$. Since these two polynomials are not multiples of each other, they are linearly independent and so they form a basis for $W^{\perp}$.