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Practical No-4
Orthogonal, orthonormal sets, Gram-Schmidt orthogonalisation

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February 11, 2022

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Orthogonality

Let V be an inner product space. Two vectors $\boldsymbol{u}, \boldsymbol{v} \in V$ are said to be **orthogonal** if

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0.$$

Example For inner product space $C[-\pi, \pi]$, the functions $\sin t$ and $\cos t$ are orthogonal as

$$\begin{aligned} \langle \sin t, \cos t \rangle &= \int_{-\pi}^{\pi} \sin t \, \cos t \, dt \\ &= \frac{1}{2} \sin^2 t \Big|_{-\pi}^{\pi} = 0 - 0 = 0. \end{aligned}$$

Example Let $\boldsymbol{u} = [a_1, a_2, \ldots, a_n]^T \in \mathbb{R}^n$. The set of all vector of the Euclidean n-space $\mathbb{R}^n$ that are orthogonal to $\boldsymbol{u}$ is a subspace of $\mathbb{R}^n$. In fact, it is the solution space of the single linear equation

$$\langle \boldsymbol{u}, \boldsymbol{x} \rangle = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0.$$

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Example Let $\boldsymbol{u} = [1,2,3,4,5]^T$, $\boldsymbol{v} = [2,3,4,5,6]^T$, and $\boldsymbol{w} = [1,2,3,3,2]^T \in \mathbb{R}^5$. The set of all vectors of $\mathbb{R}^5$ that are orthogonal to $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ is a subspace of $\mathbb{R}^5$. In fact, it is the solution space of the linear system

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 = 0 \\ x_1 + 2x_2 + 3x_3 + 3x_4 + 2x_5 = 0 \end{cases}$$

Let S be a nonempty subset of an inner product space V. We denote by $S^\perp$ the set of all vectors of V that are orthogonal to every vector of S, called the **orthogonal complement** of S in V. In notation,

$$S^\perp := \left\{ \boldsymbol{v} \in V \mid \langle \boldsymbol{v}, \boldsymbol{u} \rangle = 0 \text{ for all } \boldsymbol{u} \in S \right\}.$$

If S contains only one vector $\boldsymbol{u}$, we write

$$\boldsymbol{u}^\perp = \left\{ \boldsymbol{v} \in V \mid \langle \boldsymbol{v}, \boldsymbol{u} \rangle = 0 \right\}.$$

Orthogonal sets and bases

Let V be an inner product space. A subset $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ of nonzero vectors of V is called an **orthogonal set** if every pair of vectors are orthogonal, i.e.,

$$\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = 0, \quad 1 \leq i < j \leq k.$$

An orthogonal set $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is called an **orthonormal set** if we further have

$$\|\boldsymbol{u}_i\| = 1, \quad 1 \leq i \leq k.$$

An **orthonormal basis** of V is a basis which is also an orthonormal set.

Theorem (Pythagoras). *Let $v_1, v_2, \ldots, v_k$ be mutually orthogonal vectors. Then*

$$\|v_1 + v_2 + \cdots + v_k\|^2 = \|v_1\|^2 + \|v_2\|^2 + \cdots + \|v_k\|^2.$$

Proof. For simplicity, we assume $k = 2$. If u and v are orthogonal, i.e., $\langle u, v\rangle = 0$, then

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v\rangle \\ &= \langle u, u\rangle + \langle v, v\rangle \\ &= \|u\|^2 + \|v\|^2. \end{aligned}$$

□

Example The three vectors

$$\boldsymbol{v}_1 = [1, 2, 1]^T, \quad \boldsymbol{v}_2 = [2, 1, -4]^T, \quad \boldsymbol{v}_3 = [3, -2, 1]^T$$

are mutually orthogonal. Express the the vector $\boldsymbol{v} = [7, 1, 9]^T$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

Set

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 = \boldsymbol{v}.$$

There are two ways to find x_1, x_2, x_3.

Method 1: Solving the linear system by performing row operations to its augmented matrix

$$[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \mid \boldsymbol{v}],$$

we obtain $x_1 = 3$, $x_2 = -1$, $x_3 = 2$. So $\boldsymbol{v} = 3\boldsymbol{v}_1 - \boldsymbol{v}_2 + 2\boldsymbol{v}_3$.

Method 2: Since $\boldsymbol{v}_i \perp \boldsymbol{v}_j$ for $i \neq j$, we have

$$\langle \boldsymbol{v}, \boldsymbol{v}_i \rangle = \langle x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3, \boldsymbol{v}_i \rangle = x_i \langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle,$$

where $i = 1, 2, 3$. Then

$$x_i = \frac{\langle \boldsymbol{v}, \boldsymbol{v}_i \rangle}{\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle}, \quad i = 1, 2, 3.$$

We then have

$$\begin{aligned} x_1 &= \frac{7+2+9}{1+4+1} = \frac{18}{6} = 3, \\ x_2 &= \frac{14+1-36}{4+1+16} = \frac{-21}{21} = -1, \\ x_3 &= \frac{21-2+9}{9+4+1} = \frac{28}{14} = 2. \end{aligned}$$

Gram-Schmidt process

Let W be a subspace of an inner product space V. Let $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a basis of W, not necessarily orthogonal. An orthogonal basis $\mathcal{B}' = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\}$ may be constructed from $\mathcal{B}$ as follows:

$$
\begin{aligned}
\boldsymbol{w}_1 &= \boldsymbol{v}_1, & W_1 &= \operatorname{Span}\{\boldsymbol{w}_1\},\\
\boldsymbol{w}_2 &= \boldsymbol{v}_2 - \operatorname{Proj}_{W_1}(\boldsymbol{v}_2), & W_2 &= \operatorname{Span}\{\boldsymbol{w}_1, \boldsymbol{w}_2\},\\
\boldsymbol{w}_3 &= \boldsymbol{v}_3 - \operatorname{Proj}_{W_2}(\boldsymbol{v}_3), & W_3 &= \operatorname{Span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\},\\
&\vdots & &\\
\boldsymbol{w}_{k-1} &= \boldsymbol{v}_{k-1} - \operatorname{Proj}_{W_{k-1}}(\boldsymbol{v}_{k-1}), & W_{k-1} &= \operatorname{Span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k-1}\},\\
\boldsymbol{w}_k &= \boldsymbol{v}_k - \operatorname{Proj}_{W_{k-1}}(\boldsymbol{v}_k). & &
\end{aligned}
$$

More precisely,

$$\begin{aligned}
\boldsymbol{w}_1 &= \boldsymbol{v}_1, \\
\boldsymbol{w}_2 &= \boldsymbol{v}_2 - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_2 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1, \\
\boldsymbol{w}_3 &= \boldsymbol{v}_3 - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_3 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 - \frac{\langle \boldsymbol{w}_2, \boldsymbol{v}_3 \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle}\boldsymbol{w}_2, \\
&\vdots \\
\boldsymbol{w}_k &= \boldsymbol{v}_k - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_k \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 - \frac{\langle \boldsymbol{w}_2, \boldsymbol{v}_k \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle}\boldsymbol{w}_2 - \cdots - \frac{\langle \boldsymbol{w}_{k-1}, \boldsymbol{v}_k \rangle}{\langle \boldsymbol{w}_{k-1}, \boldsymbol{w}_{k-1} \rangle}\boldsymbol{w}_{k-1}.
\end{aligned}$$

The method of constructing the orthogonal vector $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k$ is known as the **Gram-Schmidt proces**

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Clearly, the vector $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k$ are linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$. Conversely, the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are also linear combinations of $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k$:

$$\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{w}_1, \\
\boldsymbol{v}_2 &= \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_2 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 + \boldsymbol{w}_2, \\
\boldsymbol{v}_3 &= \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_3 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 + \frac{\langle \boldsymbol{w}_2, \boldsymbol{v}_3 \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle}\boldsymbol{w}_2 + \boldsymbol{w}_3, \\
&\vdots \\
\boldsymbol{v}_k &= \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_k \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 + \frac{\langle \boldsymbol{w}_2, \boldsymbol{v}_k \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle}\boldsymbol{w}_2 + \cdots + \frac{\langle \boldsymbol{w}_{k-1}, \boldsymbol{v}_k \rangle}{\langle \boldsymbol{w}_{k-1}, \boldsymbol{w}_{k-1} \rangle}\boldsymbol{w}_{k-1} + \boldsymbol{w}_k.
\end{aligned}$$

Hence

$$\operatorname{Span}\left\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\right\} = \operatorname{Span}\left\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\right\}.$$

Since $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a basis for W, so is the set $\mathcal{B}' = \left\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\right\}$.

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Example Let W be the subspace of $\mathbb{R}^4$ spanned by

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Construct an orthogonal basis for W.

Set $\boldsymbol{w}_1 = \boldsymbol{v}_1$. Let $W_1 = \text{Span}\{\boldsymbol{w}_1\}$. To find a vector $\boldsymbol{w}_2$ in W that is orthogonal to W_1, set

$$\begin{aligned} \boldsymbol{w}_2 &= \boldsymbol{v}_2 - \text{Proj}_{W_1}\boldsymbol{v}_2 = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_2\rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle}\boldsymbol{w}_1 \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}. \end{aligned}$$

Let $W_2 = \text{Span}\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$. To find a vector $\boldsymbol{w}_3$ in W that is orthogonal to W_2, set

$$\begin{aligned} \boldsymbol{w}_3 &= \boldsymbol{v}_3 - \text{Proj}_{W_2}\boldsymbol{v}_3 \\ &= \boldsymbol{v}_3 - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_3\rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle}\boldsymbol{w}_1 - \frac{\langle \boldsymbol{w}_2, \boldsymbol{v}_3\rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2\rangle}\boldsymbol{w}_2 \end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{v}_3 - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_3\rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle}\boldsymbol{w}_1 - \frac{\langle \boldsymbol{w}_2, \boldsymbol{v}_3\rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2\rangle}\boldsymbol{w}_2 \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{2}{4}}{\frac{12}{16}} \cdot \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} \\
&= \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Then the set $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ is an orthogonal basis for W.

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We want to convert an arbitrary basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of $\mathcal{V}$ to an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$.

Idea: construct $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ successively so that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an ON basis for span$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$, $k = 1, \ldots, n$.

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Construction

$k = 1$:

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}.$$

$k = 2$: Consider the projection of $\mathbf{x}_2$ onto $\mathbf{u}_1$, i.e.,

$$\langle \mathbf{u}_1, \mathbf{x}_2 \rangle \mathbf{u}_1.$$

Then

$$\mathbf{v}_2 = \mathbf{x}_2 - \langle \mathbf{u}_1, \mathbf{x}_2 \rangle \mathbf{u}_1$$

and

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}.$$

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In general, consider $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ as a given ON basis for span$\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$.
Use the Fourier expansion to express $\boldsymbol{x}_{k+1}$ with respect to $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{k+1}\}$:

$$\boldsymbol{x}_{k+1} = \sum_{i=1}^{k+1} \langle \boldsymbol{u}_i, \boldsymbol{x}_{k+1} \rangle \boldsymbol{u}_i$$

$$\iff \quad \boldsymbol{x}_{k+1} = \sum_{i=1}^{k} \langle \boldsymbol{u}_i, \boldsymbol{x}_{k+1} \rangle \boldsymbol{u}_i + \langle \boldsymbol{u}_{k+1}, \boldsymbol{x}_{k+1} \rangle \boldsymbol{u}_{k+1}$$

$$\iff \quad \boldsymbol{u}_{k+1} = \frac{\boldsymbol{x}_{k+1} - \sum_{i=1}^{k} \langle \boldsymbol{u}_i, \boldsymbol{x}_{k+1} \rangle \boldsymbol{u}_i}{\langle \boldsymbol{u}_{k+1}, \boldsymbol{x}_{k+1} \rangle} = \frac{\boldsymbol{v}_{k+1}}{\langle \boldsymbol{u}_{k+1}, \boldsymbol{x}_{k+1} \rangle}$$

This vector, however is not yet normalized.