My Inspiration Shri. V.G. Patil Saheb
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Practical No-7
Problems under Properties of Eigen Values and Eigen Vectors.

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Contents

My Inspiration
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Problems under properties of eigen values and eigen vectors.

1. **Find the sum and product of the eigen values of the matrix** $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Solution: Sum of the eigen values = Sum of the main diagonal elements = -3

Product of the eigen values = $|A|$ = -1 (1 – 1) -1(-1 – 1) + 1(1- (-1)) = 2 + 2 = 4

2. Product of two eigen values of the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **is 16. Find the third eigen value**

Solution: Let the eigen values of the matrix be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1 \lambda_2 = 16$

We know that $\lambda_1 \lambda_2 \lambda_3 = |A|$ (Since product of the eigen values is equal to the determinant of the matrix)

$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$ = 6(9-1)+2(-6+2) +2(2-6) = 48-8-8 = 32

$$There\ fore, \lambda_1 \lambda_2 \lambda_3 = 32 \Rightarrow\ 16\lambda_3 = 32 \Rightarrow \lambda_3 = 2$$

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3. Find the sum and product of the eigen values of the matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ without finding the roots of the characteristic equation

<u>Solution</u>:We know that the sum of the eigen values = Trace of A = a + d

Product of the eigen values = $|A|$ = ad – bc

4. If 3 and 15 are the two eigen values of A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, find |A|, without expanding the determinant

Solution:Given $\lambda_1 = 3 \; and \; \lambda_2 = 15, \lambda_3 = ?$

We know that sum of the eigen values = Sum of the main diagonal elements

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$$\Rightarrow 3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$$

We know that the product of the eigen values = $|A|$

$\Rightarrow (3)(15)(0) = |A|$

$\Rightarrow |A| = 0$

5. **If 2, 2, 3 are the eigen values of A =** $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$**, find the eigen values of** A^T

Solution:By the property "A square matrix A and its transpose A^T have the same eigen values", the eigen values of A^T *are* 2,2,3

6. **Find the eigen values of A =** $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$ .Clearly, A is a lower triangular matrix. Hence, by the property "the characteristic roots of a triangular matrix are just the diagonal elements of the matrix", the eigen values of A are 2, 3, 4

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7. Two of the eigen values of A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ **are 3 and 6. Find the eigen values of** A^{-1}

Solution:Sum of the eigen values = Sum of the main diagonal elements = 3 +5+3 = 11

Given 3,6 are two eigen values of A. Let the third eigen value be k.

Then, 3 + 6 + k = 11 $\Rightarrow k = 2$

Therefore, the eigen values of A are 3, 6, 2

By the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

8. Find the eigen values of the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. Hence, form the matrix whose eigen values are $\frac{1}{6}$ $and - 1$

Solution: Let A =$\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. The characteristic equation of the given matrix is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements =$ 5 and $S_2 = |A| = -6$

Therefore, the characteristic equation is $\lambda^2 - 5\lambda - 6 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm 7}{2} = 6, -1$

Therefore, the eigen values of A are 6, -1

Hence, the matrix whose eigen values are $\frac{1}{6}$ $and - 1$ is A^{-1}

$$A^{-1} = \frac{1}{|A|} adj\ A$$

|A| = 4 -10 = - 6; adj A = $\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

Therefore, $A^{-1} = \frac{1}{-6}\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

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9. Find the eigen values of the inverse of the matrix A = $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

Solution:We know that A is an upper triangular matrix. Therefore, the eigen values of A are 2, 3, 4. Hence, by using the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

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10. Find the eigen values of A^3 given A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

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Solution:Given A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 1, 2, 3

Therefore, the eigen values of A^3 are $1^3, 2^3, 3^3$ i.e., 1,8,27

11. If 1 and 2 are the eigen values of a 2 x 2 matrix A, what are the eigen values of A^2 and A^{-1}?

Solution:Given 1 and 2 are the eigen values of A.

Therefore, 1^2 and 2^2 i.e., 1 and 4 are the eigen values of A^2 and 1 and $\frac{1}{2}$ are the eigen values of A^{-1}

12. If 1,1,5 are the eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigen values of 5A

Solution:By the property "If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A, then $k\lambda_1, k\lambda_2, k\lambda_3$ are the eigen values of kA, the eigen values of 5A are 5(1), 5(1), 5(5) ie., 5,5,25

13. Find the eigen values of A, $A^2, A^3, A^4, 3A, A^{-1}, A - I, 3A^3 + 5A^2 - 6A + 2I$ if A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 2, 5

The eigen values of A^2 are $2^2, 5^2$ i.e., 4, 25

The eigen values of A^3 are $2^3, 5^3$ i.e., 8, 125

The eigen values of A^4 are $2^4, 5^4$ i.e., 16, 625

The eigen values of 3A are 3(2), 3(5) i.e., 6, 15

The eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{5}$

A – I = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$

Since A - I is an upper triangular matrix, the eigen values of A- I are its main diagonal elements i.e., 1,4

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ *are* $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$ *and* $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$ where $\lambda_1 = 2$ *and* $\lambda_2 = 5$

First eigen value = $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$

= $3(2)^3 + 5(2)^2 - 6(2) + 2 = 24 + 20 - 12 + 2 = 34$

Second eigen value = $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$

= $3(5)^3 + 5(5)^2 - 6(5) + 2$

= $375 + 125 - 30 + 2 = 472$

14. Find the eigen values of adj A if A = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 3, 4, 1

We know that $A^{-1} = \frac{1}{|A|}\, adj\ A$

Adj A = $|A|\, A^{-1}$

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

$|A|$ =Product of the eigen values = 12

Therefore, the eigen values of adj A is equal to the eigen values of 12 A^{-1} i.e., $\frac{12}{3}, \frac{12}{4}, 12$ i.e., 4, 3, 12

Note: A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$. Here, A is an upper triangular matrix, B is a lower triangular matrix and C is a diagonal matrix. In all the cases, the elements in the main diagonal are the eigen values. Hence, the eigen values of A, B and C are 1, 4, 6

Practical No-7: Problems under Properties of Eigen Values and Eigen Vectors

15. Two eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ **are equal and they are** $\frac{1}{5}$ **times the third. Find them**

Solution:Let the third eigen value be λ_3

We know that $\lambda_1 + \lambda_2 + \lambda_3$ =2+3+2 = 7

Given $\lambda_1 = \lambda_2 = \frac{\lambda_3}{5}$

$$\frac{\lambda_3}{5} + \frac{\lambda_3}{5} + \lambda_3 = 7$$

$$\left[\frac{1}{5} + \frac{1}{5} + 1\right]\lambda_3 = 7 \Rightarrow \frac{7}{5}\lambda_3 = 7 \Rightarrow \lambda_3 = 5$$

Therefore, $\lambda_1 = \lambda_2 = 1$ and hence the eigen values of A are 1,1, 5

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16. If 2, 3 are the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, ***find the value of a***

Solution:Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$. Let the eigen values of A be 2, 3, k

We know that the sum of the eigen values = sum of the main diagonal elements

Therefore, 2 +3 +k = 2+ 2+2 = 6 $\Rightarrow k = 1$

We know that product of the eigen values = |A|

⇒2(3)(k) = |A|

$$\Rightarrow 6 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix} \Rightarrow 6 = 2(4) - 0 + 1(-2a) \Rightarrow 6 = 8 - 2a \Rightarrow 2a = 2 \Rightarrow a = 1$$