Practical No-8
Diagonalisation of matrix, orthogonal diagonalisation of symmetric matrix and application to quadratic form.

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Contents

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My Inspiration
Shri. V.G. Patil Saheb
Dr. V. S. Sonawne

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ORTHOGONAL TRANSFORMATION OF A SYMMETRIC MATRIX TODIAGONAL FORM:

Orthogonal matrices:

A square matrix A (with real elements) is said to be orthogonal if $AA^T = A^T A = I$ or $A^T = A^{-1}$

Problems:

1. **Check whether the matrix B is orthogonal. Justify. B =** $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: Condition for orthogonality is $AA^T = A^T A = I$

To prove that: $\boldsymbol{BB^T = B^T B = I}$

B = $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $B^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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$$BB^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cos^2\theta + sin^2\theta & -sin\theta cos\theta + sin\theta cos\theta & 0 \\ -sin\theta cos\theta + \sin\theta\cos\theta + 0 & sin^2\theta + cos^2\theta + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly,

$$B^T B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} cos^2\theta + sin^2\theta & \sin\theta cos\theta - \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta - \sin\theta\cos\theta & sin^2\theta + cos^2\theta + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, B is an orthogonal matrix

2. Show that the matrix P = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **is orthogonal**

Solution:To prove that: $PP^T = P^TP = I$

$$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix};\ P^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$PP^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Similarly, P^TP = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, P is an orthogonal matrix

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WORKING RULE FOR DIAGONALIZATION

[ORTHOGONAL TRANSFORMATION]:

Step 1: To find the characteristic equation

Step 2: To solve the characteristic equation

Step 3:To find the eigen vectors

Step 4: If the eigen vectors are orthogonal, then form a normalized matrix N

Step 5: Find N^T

Step 6: Calculate AN

Step 7: Calculate D = $N^T AN$

Problems:

1. **Diagonalize the matrix** $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 3 + 5 + 3 = 11$

$S_2 = \mathit{Sum\ of\ the\ minors\ of\ the\ main\ diagonalelements} = (15 - 1) + (9 - 1) + (15 - 1)$
$= 14 + 8 + 14 = 36$

$S_3 = |A| = 3(15 - 1) + 1(-3 + 1) + 1(1 - 5) = 3(14) - 2 - 4 = 42 - 6 = 36$

Therefore, the characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

2	1	-11	36	-36
	0	2	-18	36
	1	-9	18	0

$\lambda = 2, \lambda^2 - 9\lambda + 18 = 0 \Rightarrow \lambda = 2, \lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81-72}}{2} = \frac{9 \pm 3}{2} = 6,3$

Hence, the eigen values of A are 2, 3, 6

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 2$, $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 + 3x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 + x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

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$x_1 x_2 x_3$

-1 1 1 -1

3 -1 -1 3

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case 2: When $\lambda = 3$, $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 + 2x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 + 0x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

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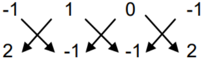
$x_1 x_2 x_3$



$$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case 3: When $\lambda = 6$, $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 - x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 - 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

Practical No-8: Diagonalisation of matrix, application to quadratic form.

$$x_1 x_2 x_3$$

-1 1 -3 -1

-1 -1 -1 -1

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$X_2^T X_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

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Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N = $\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$; $N^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

AN = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix}$

$$N^T AN = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & \frac{0}{\sqrt{6}} & \frac{0}{\sqrt{12}} \\ \frac{0}{\sqrt{6}} & \frac{9}{3} & \frac{0}{\sqrt{18}} \\ \frac{0}{\sqrt{12}} & \frac{0}{\sqrt{18}} & \frac{36}{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

The diagonal elements are the eigen values of A

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2. Diagonalize the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 8 + 7 + 3 = 18$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonalelements = (21 - 16) + (24 - 4) + (56 - 36)$
$= 5 + 20 + 20 = 45$

$S_3 = |A| = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 8(5) - 60 + 20 = 0$

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$ i.e., $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0 \Rightarrow \lambda = 0, \lambda = \frac{18 \pm \sqrt{(-18)^2 - 4(1)(45)}}{2(1)} = \frac{18 \pm \sqrt{324 - 180}}{2} = \frac{18 \pm 12}{2}$
$= 15, 3$

Hence, the eigen values of A are 0, 3, 15

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To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 0$, $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$8x_1 - 6x_2 + 2x_3 = 0$ ---------- (1)

$-6x_1 + 7x_2 - 4x_3 = 0$ -------- (2)

$2x_1 - 4x_2 + 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

x_1 x_2 x_3

-6 2 8 -6

7 -4 -6 7

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36} \Rightarrow \frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case 2: When $\lambda = 3$, $\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$5x_1 - 6x_2 + 2x_3 = 0$ -------- (1)

$-6x_1 + 4x_2 - 4x_3 = 0$ ------- (2)

$2x_1 - 4x_2 + 0x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$x_1 \quad x_2 \quad x_3$

-6 2 5 -6

4 -4 -6 4

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$$\frac{x_1}{24-8}=\frac{x_2}{-12+20}=\frac{x_3}{20-36}\Rightarrow\frac{x_1}{16}=\frac{x_2}{8}=\frac{x_3}{-16}\Rightarrow\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{-2}$$

$$X_2=\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

Case 3: When $\lambda = 15$, $\begin{bmatrix}-7 & -6 & 2\\ -6 & -8 & -4\\ 2 & -4 & -12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$-7x_1 - 6x_2 + 2x_3 = 0$ ---------- (1)

$-6x_1 - 8x_2 - 4x_3 = 0$ ---------- (2)

$2x_1 - 4x_2 - 12x_3 = 0$ ---------- (3)

Solving (1) and (2) by rule of cross-multiplication,

x_1 x_2 x_3

-6 2 -7 -6

-8 -4 -6 -8

$$\frac{x_1}{24+16}=\frac{x_2}{-12-28}=\frac{x_3}{56-36}\Rightarrow\frac{x_1}{40}=\frac{x_2}{-40}=\frac{x_3}{20}\Rightarrow\frac{x_1}{2}=\frac{x_2}{-2}=\frac{x_3}{1}$$

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$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = [1 \quad 2 \quad 2] \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2+2-4=0$$

$$X_2^T X_3 = [2 \quad 1 \quad -2] \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 4-2-2 = 0$$

$$X_3^T X_1 = [2 \quad -2 \quad 1] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2-4+2=0$$

Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N = $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$N^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

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$$N^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

AN =

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} =$$

$$\frac{1}{3}\begin{bmatrix} 8-12+4 & 16-6-4 & 16+12+2 \\ -6+14-8 & -12+7+8 & -12-14-4 \\ 2-8+6 & 4-4-6 & 4+8+3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$N^T AN = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0+0+0 & 2+2-4 & 10-20+10 \\ 0+0+0 & 4+1+4 & 20-10-10 \\ 0+0+0 & 4-2-2 & 20+20+5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 45 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

The diagonal elements are the eigen values of A

QUADRATIC FORM- REDUCTION OF QUADRATIC FORM TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION:

Quadratic form:

A homogeneous polynomial of second degree in any number of variables is called a quadratic form

Example: $2x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_2 + 5x_1x_3 - 6x_2x_3$ is a quadratic form in three variables

Note:

The matrix corresponding to the quadratic form is

$$\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$$

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Problems:

1. **Write the matrix of the quadratic form** $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$

Solution:Q = $\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$

Here $x_2x_1 = x_1x_2$; $x_3x_1 = x_1x_3$; $x_2x_3 = x_3x_2$

$$Q = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$$

2. **Write the matrix of the quadratic form** $2x^2 + 8z^2 + 4xy + 10xz - 2yz$

Solution: Q = $\begin{bmatrix} coeff.of\ x^2 & \frac{1}{2}\ coeff.of\ xy & \frac{1}{2}\ coeff.of\ xz \\ \frac{1}{2}\ coeff.of\ yx & coeff.of\ y^2 & \frac{1}{2}\ coeff.of\ yz \\ \frac{1}{2}\ coeff.of\ zx & \frac{1}{2}\ coeff.of\ zy & coeff.of\ z^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$

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3. Write down the quadratic form corresponding to the following symmetric matrix

$$\begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$$

Solution:Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

The required quadratic form is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2(a_{12})x_1x_2 + 2(a_{23})x_2x_3 + 2(a_{13})x_1x_3$$

$= 0x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 + 8x_2x_3$

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NATURE OF THE QUADRATIC FORM:

Rank of the quadratic form: The number of square terms in the canonical form is the rank (r) of the quadratic form

Index of the quadratic form: The number of positive square terms in the canonical form is called the index (s) of the quadratic form

Signature of the quadratic form: The difference between the number of positive and negative square terms = s – (r-s) = 2s-r, is called the signature of the quadratic form

The quadratic form is said to be

(1) **Positive definite** if all the eigen values are positive numbers
(2) **Negative definite** if all the eigen values are negative numbers
(3) **Positive Semi-definite** if all the eigen values are greater than or equal to zero and at least one eigen value is zero
(4) **Negative Semi-definite** if all the eigen values are less than or equal to zero and at least one eigen value is zero
(5) **Indefinite** if A has both positive and negative eigen values