Tutorial 1: Find general solution of Langrange's equation.

Subject Teacher
Santosh Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

ssdhamone@acscollegeonde.ac.in

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Langrange's Auxiliary Equation

$P(x, y, z)\ \ p + Q(x,y,z)\ \ q = R(x,y,z)$

where P, Q and R are continuously differentiable functions on the domain $D \subseteq R^3$ is $\phi(u, v) = 0$ where ϕ is an arbitrary function and

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$

are two independent solutions of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

This equation is known as Langrange's Auxiliary Equation

Problem 1:

Solve the PDE of $\frac{y^2 z}{x}\ \ p + xz\ \ q = y^2$

Solution

It is of the form

$P\ \ p + Q\ \ q = R$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \tag{1}$$

General Solution of Langrange's Equation

Solution of Problem 1 Continue...

Taking the first two fractions from (1)

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz}$$

$$x^2dx = y^2dy$$

Integrating we get, $x^3 = y^3 + c_1$

$$x^3 - y^3 = c_1 \tag{2}$$

Solution of Problem 1 Continue...

Taking the first and last fractions from (1)

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dz}{y^2}$$

$$xdx = zdz$$

Integrating we get, $x^2 = z^2 + c_2$

$$x^2 - z^2 = c_2 \tag{3}$$

General Solution of Langrange's Equation

Solution of Problem 1 Continue...

From equations (2) and (3), the required general solution is

$$\Phi(x^3 - y^3 \,,\, x^2 - z^2)$$

where ϕ is arbitrary function.

General Solution of Langrange's Equation

Problem 2:

Solve the PDE of $p + 3\ q = 5z + tan(y - 3x)$

Solution

It is of the form

$P\ \ p + Q\ \ q = R$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + tan(y - 3x)} \tag{4}$$

Solution of Problem 2 Continue...

Taking the first two fractions from (4)

$$\frac{dx}{1} = \frac{dy}{3}$$

$$3dx = ydy$$

Integrating we get, $3x = y + c_1$

$$3x - y = c_1 \tag{5}$$

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Solution of Problem 2 Continue...

Taking the first and last fractions from (4)

$$\frac{dx}{1} = \frac{dz}{5z + tan(y - 3x)}$$

$$dx = \frac{dz}{5z + tan(c_1)} \text{.....From equation (5)}$$

Integrating we get, $x = \dfrac{\ln(5z + tan\ c_1)}{5}$

$$5x - \ln(5z + tan\ c_1) = c_2 \tag{6}$$

General Solution of Langrange's Equation

Solution of Problem 2 Continue...

From equations (5) and (6), the required general solution is

$$\Phi(3x - y\ , 5x - \ln\left[5z + tan(3x - y)\right])$$

where ϕ is arbitary function.

General Solution of Langrange's Equation

Problem 3:

Solve the PDE of $yz\frac{b-c}{a}\ p + zx\frac{c-a}{b}\ q = \frac{a-b}{c}xy$

Solution

It is of the form

$$P\ \ p + Q\ \ q = R$$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{yz(b-c)}{a}} = \frac{dy}{\frac{xz(c-a)}{b}} = \frac{dz}{\frac{xy(a-b)}{c}} \tag{7}$$

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Solution of Problem 3 Continue...

Choosing x, y, z as multipliers, each fraction equals

$$=\frac{axdx}{xyz(b-c)}+\frac{bydy}{xyz(c-a)}+\frac{czdz}{xy(a-b)}$$

$$=\frac{axdx+bydy+czdz}{xyz(b-c+c-a+a-b)} \implies \text{axdx+bydy+czdz=0}$$

$$\text{Integrating we get, } \frac{ax^2}{2}+\frac{by^2}{2}+\frac{cz^2}{2}=c_1$$

$$\text{a}x^2+by^2+cz^2=c_1 \tag{8}$$

Solution of Problem 3 Continue...

Choosing ax, by, cz as multipliers, each fraction equals

$$e = \frac{a^2xdx}{xyz(b-c)} + \frac{b^2ydy}{xyz(c-a)} + \frac{c^2zdz}{xy(a-b)}$$

$$= \frac{a^2xdx + b^2ydy + c^2zdz}{xyz(ab - ac + bc - ba + ca - cb)}$$

$$\implies a^2xdx + b^2ydy + c^2zdz = 0$$

Integrating we get, $\frac{a^2x^2}{2} + \frac{b^2y^2}{2} + \frac{c^2z^2}{2} = c_2$

Solution of Problem 3 Continue...

$$a^2x^2 + b^2y^2 + c^2z^2 = c_2 \tag{9}$$

From equations (8) and (9), the required general solution is

$$\Phi(ax^2 + by^2 + cz^2 \ , \ a^2x^2 + b^2y^2 + c^2z^2)$$

where ϕ is arbitary function.

Problem 4:

Solve the PDE of $(y+z)\ p+(z+x)\ q=x+y$

Solution

It is of the form

$$P\ \ p+Q\ \ q=R$$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{y+z}=\frac{dy}{x+z}=\frac{dz}{x+y} \tag{10}$$

General Solution of Langrange's Equation

Solution of Problem 4 Continue...

Choosing $1, -1, 0$ as multipliers, each fraction equals

$$= \frac{dx - dy}{y - x} = -\frac{d(x - y)}{x - y}$$

Choosing $0, 1, -1$ as multipliers, each fraction equals

$$= \frac{dy - dz}{z - y} = -\frac{d(y - z)}{y - z}$$

Choosing $1, 1, 1$ as multipliers, each fraction equals

$$= \frac{dx + dy + dz}{y + z + z + x + x + y}$$

$$= \frac{dx + dy + dz}{2(x + y + z)}$$

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Solution of Problem 4 Continue...

Taking above three ratios,

$$-\frac{d(x-y)}{x-y} = -\frac{d(y-z)}{y-z} = \frac{dx+dy+dz}{2(x+y+z)}$$

taking the first two fraction

$$-\frac{d(x-y)}{x-y} = -\frac{d(y-z)}{y-z}$$

Integrating we get,

$$\ln x - y = \ln y - z + \ln c_1$$

$$\frac{x-y}{y-z} = c_1 \tag{11}$$

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Solution of Problem 4 Continue...

Taking First and Third ratios,

$$-\frac{d(x-y)}{x-y}=\frac{dx+dy+dz}{2(x+y+z)}$$

Integrating we get,

$$-2\ln x-y=\ln(x+y+z)+\ln c_2$$

$$\ln(x-y)^2+\ln(x+y+z)=-\ln c_2$$

$$\ln(x-y)^2(x+y+z)=\ln c_2$$

Solution of Problem 4 Continue...

$$(x+y+z)(x\text{-}y)^2 = c_2 \tag{12}$$

From equations (11) and (12), the required general solution is

$$\Phi\left(\frac{x-y}{y-z}, (x+y+z)(x-y)^2\right)$$

where ϕ is arbitary function.

Problem 5:

Solve the PDE of $(x+2z)\ p+(4xz-y)\ q=2x^2+y$

Solution

It is of the form

$$P\ p+Q\ q=R$$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{(x+2z)}=\frac{dy}{(4xz-y)}=\frac{dz}{(2x^2+y)} \tag{13}$$

Solution of Problem 5 Continue...

Choosing $-2x, 1, 1$ as multipliers, each fraction equals

$$= \frac{-2xdx + dy + dz}{-2x^2 - 4xz + 4xz - y + 2x^2 + y}$$

∴ -2xdx+dy+dz=0

Integrating we get,

$$\therefore \quad \frac{-2x^2}{2} + y + z = c_1$$

$$-x^2 + y + z = c_1 \tag{14}$$

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Solution of Problem 5 Continue...

Choosing $y, x, -2z$ as multipliers, each fraction equals

$$=\frac{ydx + xdy - 2zdz}{xy + 2yz + 4x^2z - xy - 4x^2z - 2yz}$$

$$\therefore \text{ ydx+xdy-2zdz=0}$$

$$\therefore \text{ d(xy)-2zdz=0}$$

$$\text{Integrating we get, } xy + \frac{-2z^2}{2} = c_2$$

$$\text{xy-z}^2 = c_2 \qquad (15)$$

From equations (14) and (15), the required general solution is

$$\Phi(\text{-x}^2 + y + z \ , \ xy - z^2)$$

where ϕ is arbitary function.

Problem 6:

Solve the PDE of $(y - zx)\ p + (x + yz)\ q = x^2 + y^2$

Solution

It is of the form

$$P\ \ p + Q\ \ q = R$$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{(y - zx)} = \frac{dy}{(x + yz)} = \frac{dz}{(x^2 + y^2)} \tag{16}$$

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Solution of Problem 6 Continue...

Choosing $y, x, -1$ as multipliers, each fraction equals

$$=\frac{ydx + xdy - dz}{y^2 - xyz + x^2 + xyz - x^2 - y^2}$$

$\therefore$ ydx+xdy-dz=0$\therefore$ d(xy)-dz=0$Integratingweget,$

$\therefore$ xy - z= c_1

$$\text{xy - z} = c_1 \tag{17}$$

Solution of Problem 6 Continue...

Choosing $x, -y, z$ as multipliers, each fraction equals

$$= \frac{xdx - ydy + zdz}{xy - x^2z - xy - y^2z + x^2z + y^2z}$$

$$\therefore \text{xdx - ydy} + \text{zdz} = 0$$

$$\text{Integrating we get, } \frac{x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$x^2 - y^2 + z^2 = c_2 \tag{18}$$

From equations (17) and (18), the required general solution is

$$\Phi(\text{xy - z}, x^2 - y^2 + z^2)$$

where ϕ is arbitary function.

Problem 7:

Solve the PDE of $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$

Solution

Let $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$

It is of the form

$$P\ p + Q\ q = R$$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \tag{19}$$

Solution of Problem 7 Continue...

Consider first two ratio of equation (19)

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\therefore x^{-2}dx = y^{-2}dy$$

Integrating we get,

$$\therefore \quad \frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + c_1$$

$$\therefore \quad \frac{1}{y} - \frac{1}{x} = c_1$$

$$\frac{x-y}{xy} = c_1 \tag{20}$$

General Solution of Langrange's Equation

Solution of Problem 7 Continue...

Choosing $\frac{1}{x}, \frac{1}{y}, -\frac{1}{z}$ as multipliers, each fraction equals

$$=\frac{\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{x + y - x - y} \implies \frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz = 0$$

Integrating we get,

$$\therefore \ \ln x + \ln y - \ln z = \ln c_2 \implies \therefore \ \ln \frac{xy}{z} = \ln c_2$$

$$\frac{xy}{z} = c_2 \qquad (21)$$

From equations (20) and (21), the required general solution is

$$\Phi\left(\frac{x-y}{xy} , \frac{xy}{z} \right)$$

General Solution of Langrange's Equation

Problem 8:

Find the general solution / complete integral of PDE
$z(x\ p - y\ q) = y^2 - x^2$

Solution

Given $xz\ p - yz\ q = y^2 - x^2$
It is of the form $P\ p + Q\ q = R$
Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{xz} = \frac{dy}{-yz} = \frac{dz}{y^2 - x^2} \tag{22}$$

Solution of Problem 8 Continue...

Consider first two ratio of equation (22)

$$\frac{dx}{xz} = \frac{dy}{-yz}$$

$$\therefore \quad \frac{dx}{x} = \frac{dy}{-y}$$

Integrating we get,

$$\therefore \ln x = -\ln y + \ln c_1$$

$$\therefore \ln x + \ln y = \ln c_1$$

$$\therefore \quad \ln xy = \ln c_1$$

Taking antilog, we get

$$xy = c_1 \tag{23}$$

Solution of Problem 8 Continue...

Choosing $(x+y),(x+y),z$ as multipliers, fraction equals

$$= \frac{(x+y)dx + (x+y)dy + zdz}{x^2z + xyz - xyz - y^2z + y^2z - x^2z}$$

$$xdx + ydy + (ydy + xdy) + zdz = 0$$

Integrating we get,

$$\frac{x^2}{2} + \frac{y^2}{2} + xy + \frac{z^2}{2} = c_2$$

$$x^2 + y^2 + 2xy + z^2 = c_2 \tag{24}$$

From equations (23) and (24), the required general solution is

$$\Phi(xy, x^2 + y^2 + 2xy + z^2)$$

where ϕ is arbitary function.

Problem 9:

Find the general solution / complete integral of PDE
$y^2 \; p - xy \; q = x(z - 2y)$

Solution

Given $y^2 \; p - xy \; q = x(z - 2y)$
It is of the form $P \; p + Q \; q = R$
Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \tag{25}$$

Solution of Problem 9 Continue...

Consider first two ratio of equation (25)

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\therefore \quad \frac{dx}{y} = \frac{dy}{-x}$$

$\therefore$ xdx = -ydy Integrating we get,

$$\therefore \quad \frac{x^2}{2} = -\frac{y^2}{2} + c_1$$

$$\therefore \quad \frac{x^2}{2} + \frac{y^2}{2} = c_1$$

Taking antilog, we get

$$x^2 + y^2 = c_1 \tag{26}$$

Solution of Problem 9 Continue...

Choosing $2x, z, y$ as multipliers, fraction equals

$$= \frac{2xdx + zdy + ydz}{2xy^2 - xyz + xyz - 2xy^2}$$

$$\therefore 2\mathrm{xdx} + \mathrm{zdy} + \mathrm{ydz} = 0 \Longrightarrow 2\mathrm{xdx} + \mathrm{d(yz)} = 0$$

Integrating we get,

$$2\frac{x^2}{2} + yz = c_2$$

$$\mathrm{x}^2 + 2yz = c_2 \tag{27}$$

From equations (26) and (27), the required general solution is

$$\Phi(\ \mathrm{x}^2 + y^2\ ,\ x^2 + 2yz)$$

where ϕ is arbitary function.

General Solution of Langrange's Equation

Problem 10:

Solve the PDE of $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x + y + z$

Solution

$$\text{Given } p + q = x + y + z$$

$$\text{It is of the form} \quad P \;\; p + Q \;\; q = R$$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x + y + z} \tag{28}$$

Solution of Problem 10 Continue...

Consider first two ratio of equation (28)

$$\frac{dx}{1} = \frac{dy}{1}$$

$$\therefore \mathrm{dx} = \mathrm{dy}$$

Integrating we get,

$$\therefore \mathrm{x} = \mathrm{y} + \mathrm{c}_1$$

$$\mathrm{x} - \mathrm{y} = \mathrm{c}_1 \qquad (29)$$

Solution of Problem 10 Continue...

Choosing 1, 1, 1 as multipliers, fraction equals

$$\frac{dx}{1} = \frac{dx + dy + dz}{2 + x + y + z}$$

Integrating we get, $x = \ln(2 + x + y + z) + \ln c_2$

$$x = \ln[c_2(2 + x + y + z)] \implies e^x = c_2(2 + x + y + z)$$

$$\frac{e^x}{(2 + x + y + z)} = c_2 \tag{30}$$

From equations (28) and (29), the required general solution is

$$\Phi\left(x - y , \frac{e^x}{(2 + x + y + z)}\right)$$

where ϕ is arbitary function.