UNIT II : Compatible System of First Order Partial Differential Equations

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Compatible Differential Equations

Compatible Differential Equations

Let $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ be first order partial differentiable equations.If every solution of $f = 0$ is also solution of $g = 0$ and

Jacobian $J = \dfrac{\partial(f, g)}{\partial(p, q)} \neq 0$

then this two equations f and g are said to be Compatiable.

Compatible Differential Equations

Theorem :

Show that the condition for $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ compatiable is $[f, g] = 0$
i.e. $\dfrac{\partial(f, g)}{\partial(x, p)} + \dfrac{\partial(f, g)}{\partial(y, q)} + p\dfrac{\partial(f, g)}{\partial(z, p)} + q\dfrac{\partial(f, g)}{\partial(z, q)} = 0$

Proof

Let

$$f(x, y, z, p, q) = 0 \tag{1}$$

$$g(x, y, z, p, q) = 0 \tag{2}$$

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Proof of Theorem Continue...

be first order partial differential equations. From (1) and (2) we obtain

$$p = \Phi(x, y, z), \quad q = \Psi(x, y, z)$$

The condition that equations (1) and (2) should be compatible reduces to $p\ dx + q\ dy = dz$ is integrable.

$$\therefore \quad \Phi\ dx + \Psi\ dy - dz = 0 \qquad (3)$$

is integrable.

Let $\bar{X} = (\Phi, \Psi, -1)$ then $\bar{X}.Curl\bar{X} = 0$

Proof of Theorem Continue...

$$\text{Now, } Curl\bar{X} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \Phi & \Psi & -1 \end{vmatrix}$$

$$= (0 - \frac{\partial \Psi}{\partial z})\hat{i} - (0 - \frac{\partial \Phi}{\partial z})\hat{j} + (\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y})\hat{k}$$

$$= -\frac{\partial \Psi}{\partial z}\hat{i} + \frac{\partial \Phi}{\partial z}\hat{j} + (\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y})\hat{k}$$

$$\bar{X}.Curl\bar{X} = (\Phi\hat{i}, \Psi\hat{j}, -1\hat{k}).[-\frac{\partial \Psi}{\partial z}\hat{i} + \frac{\partial \Phi}{\partial z}\hat{j} + (\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y})\hat{k}]$$

$$-\Phi\frac{\partial \Psi}{\partial z} + \Psi\frac{\partial \Phi}{\partial z} - \frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y} = 0$$

Compatible Differential Equations

Proof of Theorem Continue...

$$\Psi \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial y} = \Phi \frac{\partial \Psi}{\partial z} + \frac{\partial \Psi}{\partial x} \tag{4}$$

Differentiate (1) w.r.t x and z,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} . \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} . \frac{\partial q}{\partial x} = 0$$

$$\therefore \quad f_x + f_p \Phi_x + f_q \Psi_x = 0 \tag{5}$$

and $$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p} . \frac{\partial p}{\partial z} + \frac{\partial f}{\partial q} . \frac{\partial q}{\partial z} = 0$$

Proof of Theorem Continue...

$$\therefore \quad f_z + f_p\Phi_z + f_q\Psi_z = 0 \tag{6}$$

Multiply equation (6) by Φ then add it to equation (5)

$$(f_x + \Phi f_z) + f_p(\Phi_x + \Phi\Phi_z) + f_q(\Psi_x + \Phi\Psi_z) = 0 \tag{7}$$

Differentiate (2) w.r.t x and z and as above, we get,

$$(g_x + \Phi g_z) + g_p(\Phi_x + \Phi\Phi_z) + g_q(\Psi_x + \Phi\Psi_z) = 0 \tag{8}$$

Compatible Differential Equations

Proof of Theorem Continue...

Multiply equation (7) by g_p and (8) by f_p then take (7)-(8)

$$g_p(f_x + \Phi f_z) + g_p f_p(\Phi_x + \Phi\Phi_z) + g_p f_q(\Psi_x + \Phi\Psi_z) - f_p(g_x + \Phi g_z) - f_p g_p(\Phi_x + \Phi\Phi_z) - f_p g_q(\Psi_x + \Phi\Psi_z) = 0$$

$$g_p(f_x + \Phi f_z) + g_p f_q(\Psi_x + \Phi\Psi_z) - f_p(g_x + \Phi g_z) - f_p g_q(\Psi_x + \Phi\Psi_z) = 0$$

$$g_p(f_x + \Phi f_z) - f_p(g_x + \Phi g_z) + (\Psi_x + \Phi\Psi_z)(g_p f_q - f_p g_q) = 0$$

$$\Phi(g_p f_z - f_p g_z) + (f_x g_p - g_x f_p) + (\Psi_x + \Phi\Psi_z)(g_p f_q - f_p g_q) = 0$$

$$(f_x g_p - g_x f_p) + \Phi(g_p f_z - f_p g_z) = (\Psi_x + \Phi\Psi_z)(f_p g_q - (g_p f_q)$$

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)} + \Phi\frac{\partial(f,g)}{\partial(z,p)} = J(\Psi_x + \Phi\Psi_z)$$

Compatible Differential Equations

Proof of Theorem Continue...

$$\therefore \quad (\Psi_x + \Phi\Psi_z) = \frac{1}{J} \quad \left[\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)}\right] \qquad (9)$$

Similarly diff. eq^{ns} (1) and (2) w.r.t y and z, we obtain

$$\therefore \quad (\Phi_y + \Psi\Phi_z) = \frac{-1}{J} \quad \left[\frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}\right] \qquad (10)$$

Using equation (4) we get,

$$\frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q} = 0$$

$[f,g] = 0$ It is condition for f and g are to be compatible.

Compatible Differential Equations

Problem 1:

Show that the PDE $xp = yq$ and $z(x\,p + y\,q) = 2xy$ are compatible. Find Solution

Solution:

Let

$$f = x\ p - y\ q = 0 \tag{11}$$

$$g = z(xp + yq) - 2xy = 0 \tag{12}$$

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Solution of Problem 1 Continue...

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \frac{\partial f}{\partial x}\ \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\ \frac{\partial g}{\partial x}$$

$$= p(zx) - x(zp - 2y) = zpx - zpx + 2xy = 2xy$$

$$\frac{\partial(f,g)}{\partial(x,p)} = 2xy$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq - 2x & zy \end{vmatrix} = -2xy$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -2xy$$

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Solution of Problem 1 Continue...

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial p}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p}\end{vmatrix}=\begin{vmatrix}0 & x\\ xp+yq & zx\end{vmatrix}=-x(xp+yq)$$

$$\frac{\partial(f,g)}{\partial(z,p)}=-x(xp+yq)$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial q}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q}\end{vmatrix}=\begin{vmatrix}0 & -y\\ xp+yq & zy\end{vmatrix}=y(xp+yq)$$

$$\frac{\partial(f,g)}{\partial(z,q)}=y(xp+yq)$$

Compatible Differential Equations

Solution of Problem 1 Continue...

Condition for Compatible is

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q}$$

$$= 2xy - 2xy + p[-x(xp+yq)] + qy(xp+yq)$$

$$= -x^2p^2 - xypq + xypq + y^2q^2$$

$$= y^2q^2 - x^2p^2$$

$$[f,g] = 0$$

$\therefore$ f and g satisfies the condition of Compatibility.

$\therefore$ Given PDE are compatible.

Solution of Problem 1 Continue...

By equation (11) $xp = yq$

Using this in (12) $z(xp + xp) = 2xy$

$2xpz = 2xy \implies p = \frac{y}{z}$

Using value of p in (11), we get

$x(\frac{y}{z}) = yq \implies q = \frac{x}{z}$

Using p and q in $p\ dx + q\ dy = dz$

$\therefore \frac{y}{z}dx + \frac{x}{z}dy = dz \implies ydx + x\, dy = zdz$

$\therefore d(xy) = zdz$

Integrating, we get

$xy = \frac{z^2}{2} + c$

$2xy - z^2 = c$ .... Required Solution.

Compatible Differential Equations

Problem 2:

Show that the PDE $xp - yq = x$ and $x^2 p + q = xz$ are compatible. Hence find Solution

Solution:

Let

$$f = x\ p - y\ q - x = 0 \tag{13}$$

$$g = x^2 p + q - xz = 0 \tag{14}$$

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Solution of Problem 2 Continue...

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix} = x^2(p-1) - x(2xp-z)$$

$$\frac{\partial(f,g)}{\partial(x,p)} = x^2(p-1) - x(2xp-z)$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix} = -q$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -q$$

Solution of Problem 2 Continue...

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix} = x^2$$

$$\frac{\partial(f,g)}{\partial(z,p)} = x^2$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix} = -xy$$

$$\frac{\partial(f,g)}{\partial(z,q)} = -xy$$

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Solution of Problem 2 Continue...

Condition for Compatible is

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q}$$

$$= x^2(p-1) - x(2xz - z) + px^2 - q - qxy$$

$$=\text{-}x^2p - x^2 - 2x^2p + xz + x^2p - q - qxy$$

$$=(xz - q) - x^2 - qxy$$

$$= x^2p - x^2 - qxy \text{ .... by equation (14)}$$

$$= x(xp - yq) -x^2$$

$$= x.x - x^2$$

$$[f,g] = 0$$

$\therefore$ f and g satisfies the condition of Compatibility.

$\therefore$ Given PDE are compatible.

Solution of Problem 2 Continue...

Multiply equation (14) by y then add it in equation (13)

$$(x + x^2y)\,p = x + xyz \implies x(1 + xy) \quad p = x\,(1 + yz)$$

$$p = \frac{1 + yz}{1 + xy}$$

Using it in (13)

$$\frac{1 + yz}{1 + xy} - yq = x \implies \frac{1 + yz}{1 + xy} - x = yq$$

$$\implies yq = \frac{x + xyz - x - x^2y}{1 + xy} \implies yq = \frac{y(xz - x^2)}{1 + xy}$$

$$q = \frac{x(z - x)}{1 + xy}$$

Solution of Problem 2 Continue...

Using p and q in $p\ dx + q\ dy = dz$

$$\therefore \quad \frac{1+yz}{1+xy}dx + \frac{x(z-x)}{1+xy}dy = dz$$

It is Pfaffian Differential Equation

Take x = constant $\implies dx = 0$

$\therefore$ (xz - x^2)$dy - (1+xy)dz = 0$

$\therefore$ x(z - x) dy - (1 + xy) dz = 0

Dividing throughout by $(z-x)(1+xy)$

$$\therefore \quad \frac{x}{1+xy}dy - \frac{dz}{z-x} = 0$$

Integrating, we get

$\ln(1+xy) - \ln(z-x) = \ln c_1$

Compatible Differential Equations

Solution of Problem 2 Continue...

$\therefore \quad \dfrac{1+xy}{z-x} = c_1$ Hence Solution is of the form

$$\frac{1+xy}{z-x} = \Phi(x)$$

Hence Required Solution is

$$\frac{1+xy}{z-x} = c$$

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Charpit's Auxiliary Equation

Show that the Charpit's Auxiliary Equation is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

Proof

Let

$$f(x, y, z, p, q) = 0 \tag{15}$$

be first order partial differential equation where x , y are independent and z is dependent variable.

Proof of Charpit's Auxiliary Equation...

$$\therefore z = z\,(x\,,\,y)$$

$$\therefore dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\therefore \quad dz = p \quad dx + q \quad dy \tag{16}$$

In this method we consider a another relation

$$F(x, y, z, p, q) = 0 \tag{17}$$

such that values of p and q obtained from (15) and (17) makes equation (16) integrable.

Proof of Charpit's Auxiliary Equation...

The solution of (16) is complete integral of (15).

Differentiate (15) and (17) w.r.t. x

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0 \qquad (18)$$

Similarly,

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + \frac{\partial F}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}.\frac{\partial q}{\partial x} = 0 \qquad (19)$$

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Proof of Charpit's Auxiliary Equation...

Multiply to (18) by $\dfrac{\partial F}{\partial p}$ and (19) by $\dfrac{\partial f}{\partial p}$ then take (18) -(19)

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0$$

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + p\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right) + \frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right) = 0 \quad (20)$$

Charpit's Auxiliary Equation

Proof of Charpit's Auxiliary Equation...

Differentiate (15) and (17) w.r.t. y, we get
[In (20) replace x = y , p = q , q = p]

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + q\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right) + \frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right) = 0 \tag{21}$$

$$\text{Now,} \frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y} \implies \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$$

Proof of Charpit's Auxiliary Equation...

Adding equations (20) and (21), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)=0$$

$$\therefore \frac{\partial F}{\partial x}(-\frac{\partial f}{\partial p})+\frac{\partial F}{\partial y}(-\frac{\partial f}{\partial q})+\frac{\partial F}{\partial z}(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q})+\frac{\partial F}{\partial p}(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z})+\frac{\partial F}{\partial q}(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z})=0$$

Charpit's Auxiliary Equation

Proof of Charpit's Auxiliary Equation...

It's Auxiliary Equation is

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z}$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

It is known as Charpit's Auxiliary equation.
Finding expression for p and q from (15) and Charpit's Auxiliary equation putting this value in (16) and on integration, we get required result.

Solution of PDE by using Charpit's Method

Problem 1:

Solve the PDE $(p^2 + q^2)y = qz$ by Charpit's method

Solution

Let

$$f = (\ p^2 + q^2)y - qz = 0 \tag{22}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$

$$\frac{\partial f}{\partial y} = f_y = p^2 + q^2$$

Solution of Problem 1 Continue...

$$\frac{\partial f}{\partial z} = f_z = -q$$
$$\frac{\partial f}{\partial p} = f_p = 2py$$
$$\frac{\partial f}{\partial q} = f_q = 2qy - z$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$
$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + 2q^2y - qz} = \frac{-dp}{0 + (-pq)} = \frac{-dq}{p^2 + q^2 - q^2}$$

Solution of Problem 1 Continue...

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + 2q^2y - qz} = \frac{-dp}{-pq} = \frac{-dq}{p^2}$$

Consider two last ratios

$$\frac{dp}{pq} = \frac{-dq}{p^2} \implies \frac{dp}{q} = \frac{-dq}{p} \implies \text{pdp} = \text{-qdq}$$

Integrating,

$$\frac{p^2}{2} + \frac{q^2}{2} = a \implies \text{p}^2 + q^2 = a$$

Using $p^2 + q^2 = a$ in equation (22)

$$ay = qz \implies q = \frac{ay}{z}$$

Using $q = \frac{ay}{z}$ in $p^2 + q^2 = a$

Solution of Problem 1 Continue...

$$p^2 + \left(\frac{ay}{z}\right)^2 = a \implies p^2 = a - \frac{a^2y^2}{z^2}$$

$$\implies p^2 = \frac{az^2 - a^2y^2}{z^2} \implies p = \frac{\sqrt{az^2 - a^2y^2}}{z}$$

Consider, $p\ dx + q\ dy = dz$

$$\therefore \frac{\sqrt{az^2 - a^2y^2}}{z}\ dx + \frac{ay}{z} dy = dz$$

$$\sqrt{az^2 - a^2y^2}\ dx + \text{ay dy} = \text{z dz}$$

$$\sqrt{a}\left(\sqrt{z^2 - ay^2}\right)\ dx = \text{z dz - ay dy}$$

$$\sqrt{a}\ dx = \frac{zdz - aydy}{\sqrt{z^2 - ay^2}}$$

Integrating we get,

$\sqrt{a}\ x = \sqrt{z^2 - ay^2} + b$ ...Required Solution.

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Problem 2:

Solve the PDE $p = (z + qy)^2$ by Charpit's method

Solution

Let

$$f = p - (z + qy)^2 = 0 \tag{23}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$

$$\frac{\partial f}{\partial y} = f_y = -2(z + qy).q$$

Solution of Problem 2 Continue...

$$\frac{\partial f}{\partial z} = f_z = -2(z+qy)$$

$$\frac{\partial f}{\partial p} = f_p = 1$$

$$\frac{\partial f}{\partial q} = f_q = -2(z+qy).y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{1} = \frac{dy}{-2y(z+qy)} = \frac{dz}{p - 2yq(z+qy)} =$$

$$\frac{-dp}{-2p(z+qy)} = \frac{-dq}{-2q(z+qy) - 2q(z+qy)}$$

Solution of Problem 2 Continue...

$$\text{Consider} \quad \frac{dy}{-2y(z+qy)} = \frac{-dp}{-2p(z+qy)}$$

$$\therefore \frac{dy}{-y} = \frac{dp}{p}$$

Integrating,

$$-\ln y = \ln p - \ln a \implies \ln y + \ln p = \ln a \implies \text{yp} = \text{a}$$

$$\therefore p = \frac{a}{y}$$

Using $p = \frac{a}{y}$ in equation (23)

$$\frac{a}{y} = (z+qy)^2 \implies \text{z} + \text{qy} = \sqrt{\frac{a}{y}} \implies \text{qy} = \sqrt{\frac{a}{y}} - z$$

Solution of Problem 2 Continue...

$$\therefore\ q = \frac{1}{y}\left(\sqrt{\frac{a}{y}} - z\right)$$

Consider, $p\ dx + q\ dy = dz$

$$\therefore\ \frac{a}{y}\ dx + \ \frac{1}{y}\left(\sqrt{\frac{a}{y}} - z\right) dy = dz$$

$$adx + \sqrt{\frac{a}{y}}dy = ydz + zdy$$

$$a\ dx + \sqrt{\frac{a}{y}}dy = d(yz)$$

Integrating we get,

$$a\,x + 2\sqrt{ay} = yz + b$$

$$ax + 2\sqrt{ay} - yz = b$$

This is Required Complete Integral.

Problem 3:

Solve the PDE $2xz - px^2 - 2qxy + pq = 0$ by Charpit's method

Solution

Let

$$f = 2xz - px^2 - 2qxy + pq = 0 \tag{24}$$

$$\frac{\partial f}{\partial x} = f_x = 2z - 2px - 2qy$$
$$\frac{\partial f}{\partial y} = f_y = -2qx$$

Solution of Problem 3 Continue...

$$\frac{\partial f}{\partial z} = f_z = 2x$$
$$\frac{\partial f}{\partial p} = f_p = x^2 + q$$
$$\frac{\partial f}{\partial q} = f_q = p - 2xy$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$
$$\frac{dx}{x^2+q} = \frac{dy}{p-2xy} = \frac{dz}{p(x^2+q)+q(p-2xy)} = \frac{-dp}{2(z-px-qy)+2px} = \frac{-dq}{-2qx+2qx}$$
$$\frac{dx}{x^2+q} = \frac{dy}{p-2xy} = \frac{dz}{px^2+2pq-2qxy} = \frac{-dp}{2z-2qy} = \frac{dq}{0}$$

Solution of Problem 3 Continue...

Consider,

$$\text{Each Ratio} = \frac{dq}{0}$$

$$\therefore \text{dq} = 0$$

Integrating,we get

$$\therefore \quad q = a$$

Using $q = a$ in equation (24)

$$2xz - px^2 - 2axy + pa = 0$$

$$\therefore \text{p(a - x}^2) = 2axy - 2xz$$

$$\therefore \quad \text{p} = \frac{2x(ay - z)}{a - x^2}$$

Solution of Problem 3 Continue...

Consider, $p\ dx + q\ dy = dz$

$$\therefore \frac{2x(ay-z)}{a-x^2}\ dx + \text{a}\ \ \text{dy} = \text{dz}$$

Divide throughout by $(ay-z)$, we get,

$$\therefore \frac{2x}{a-x^2}\ dx + \frac{a}{(ay-z)}\ dy = \frac{dz}{(ay-z)}$$

$$\therefore \text{-}\ \frac{-2x}{a-x^2}\ dx + \frac{ady-dz}{(ay-z)} = 0$$

Integrating we get,

$$-\ln(a-x^2) + \ln(ay-z) = \ln b$$

$$\ln\frac{ay-z}{a-x^2} = \ln b$$

$\frac{ay-z}{a-x^2} = b$.... This is Required Complete Integral.

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Problem 4:

Solve the PDE $2(z+xp+yq) = yp^2$ by Charpit's method

Solution

Let

$$f = 2(z + xp + yq) - yp^2 = 0 \tag{25}$$

$$\frac{\partial f}{\partial x} = f_x = 2p$$

$$\frac{\partial f}{\partial y} = f_y = 2q - p^2$$

Solution of PDE by using Charpit's Method

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Solution of Problem 4 Continue...

$$\frac{\partial f}{\partial z} = f_z = 2$$

$$\frac{\partial f}{\partial p} = f_p = 2x - 2yp$$

$$\frac{\partial f}{\partial q} = f_q = 2y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{2x - 2yp} = \frac{dy}{2y} = \frac{dz}{2xp - 2yp^2 + 2yq} = \frac{-dp}{2p + 2p} = \frac{-dq}{2q - p^2 + 2q}$$

$$\frac{dx}{2x - 2yp} = \frac{dy}{2y} = \frac{dz}{2xp - 2yp^2 + 2yq} = \frac{-dp}{4p} = \frac{-dq}{4q - p^2}$$

Solution of Problem 4 Continue...

Consider,

$$\frac{dy}{2y} = \frac{-dp}{4p}$$

Integrating,we get

$$2\ln y = -\ln p + \ln a$$

$$\ln y^2 + \ln p = \ln a \implies \ln y^2 p = \ln a$$

$$\therefore \quad p = \frac{a}{y^2}$$

Using $p = \dfrac{a}{y^2}$ in equation (25)

$$2z + 2x\frac{a}{y^2} + 2yq - y(\frac{a}{y^2})^2 = 0$$

$$\therefore \quad 2z + \frac{2ax}{y^2} + 2yq - (\frac{a^2}{y^3}) = 0$$

Solution of Problem 4 Continue...

$$\therefore \quad q = \frac{a^2}{2y^4} - \frac{z}{y} - \frac{xa}{y^3}$$

Consider, $p \; dx + q \; dy = dz$

$$\therefore \quad \frac{a}{y^2} \; dx + \left(\frac{a^2}{2y^4} - \frac{z}{y} - \frac{xa}{y^3} \right) \; dy = dz$$

$$\therefore \quad \frac{a}{y^2} \; dx + \frac{a^2}{2y^4} dy - \frac{z}{y} dy - \frac{xa}{y^3} dy = dz$$

$$a \left(\frac{ydx - xdy}{y^3} \right) + \frac{a^2}{2} \frac{dy}{y^3} = ydz + zdy$$

$$ad(\frac{x}{y}) + \frac{a^2}{2} \frac{dy}{y^3} = d(yz)$$

Integrating we get,

Solution of PDE by using Charpit's Method

Solution of Problem 4 Continue...

Integrating we get,

$$a\frac{x}{y} - \frac{a^2}{4y^2} = yz + b$$

$$\frac{ax}{y} - \frac{a^2}{4y^2} - yz = b$$

This is Required Complete Integral.

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Problem 5:

Solve the PDE $pxy + pq + qy = yz$ by Charpit's method

Solution

Let

$$\text{f} = \text{pxy} + \text{pq} + \text{qy - yz} = 0 \tag{26}$$

$$\frac{\partial f}{\partial x} = f_x = py$$
$$\frac{\partial f}{\partial y} = f_y = q - z$$

Solution of Problem 5 Continue...

$$\frac{\partial f}{\partial z} = f_z = -y$$

$$\frac{\partial f}{\partial p} = f_p = xy + q$$

$$\frac{\partial f}{\partial q} = f_q = p + y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{xy+q} = \frac{dy}{p+y} = \frac{dz}{xyp+qp+pq+yq} = \frac{-dp}{py-yp} = \frac{-dq}{q-z-qy}$$

$$\frac{dx}{xy+q} = \frac{dy}{p+y} = \frac{dz}{xyp+2pq+yq} = \frac{-dp}{0} = \frac{-dq}{q-z-qy}$$

Solution of Problem 5 Continue...

Consider,

$$EachRatio = \frac{dp}{0}$$

$$\therefore \text{dp} = 0$$

Integrating,we get

$$\therefore \quad p = a$$

Using $p = a$ in equation (26)

$$axy + aq + qy - yz = 0$$

$$q(a + y) = yz - axy$$

$$\therefore \quad q = \frac{y(z - ax)}{a + y}$$

Solution of Problem 5 Continue...

Consider, $p\ dx + q\ dy = dz$

$$\therefore\quad a\ \ dx + \frac{y(z-ax)}{a+y}\,dy = dz$$

$$\therefore\quad \frac{y(z-ax)}{a+y}\,dy = dz - a\ dx$$

$$\therefore\quad \frac{y}{a+y}\,dy = \frac{dz-adx}{(z-ax)}$$

$$\therefore\quad \frac{y+a-a}{a+y}\,dy = \frac{dz-adx}{(z-ax)}$$

$$\therefore\quad \left(1-\frac{a}{a+y}\right)\,dy = \frac{dz-adx}{(z-ax)}$$

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Solution of Problem 5 Continue...

Integrating we get,

$$y - a\ln(a+y) = \ln(z-ax) + b$$

$$y - \ln(a+y)^a - \ln(z-ax) = b$$

$$y - [\ln(a+y)^a + \ln(z-ax)] = b$$

$$y - \ln(a+y)^a(z-ax) = b$$

$$y - b = \ln(a+y)^a(z-ax)$$

$$(a+y)^a(z-ax) = e^{y-b}$$

$$(a+y)^a(z-ax) = ce^y$$

This is Required Complete Integral.

Problem 6:

Solve the PDE $p^2x + q^2y = z$ by Charpit's method

Solution

Let

$$f = p^2x + q^2y - z = 0 \tag{27}$$

$$\frac{\partial f}{\partial x} = f_x = p^2$$
$$\frac{\partial f}{\partial y} = f_y = q^2$$

Solution of PDE by using Charpit's Method

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Solution of Problem 6 Continue...

$$\frac{\partial f}{\partial z} = f_z = -1$$
$$\frac{\partial f}{\partial p} = f_p = 2px$$
$$\frac{\partial f}{\partial q} = f_q = 2qy$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$
$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2p^2x + 2q^2y} = \frac{-dp}{p^2 - p} = \frac{-dq}{q^2 - q}$$

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Solution of Problem 6 Continue...

Consider,

$$\text{Each Ratio} = \frac{p^2dx + 2pxdp}{2p^2x}$$

Similarly,

$$\text{Each Ratio} = \frac{q^2dy + 2qydq}{2q^2y}$$

$$\frac{p^2dx + 2pxdp}{2p^2x} = \frac{q^2dy + 2qydq}{2q^2y}$$

Integrating,we get

$$\ln p^2x = \ln q^2y + \ln a$$

$$\ln p^2x = \ln aq^2y \implies \mathrm{p}^2x = aq^2y \implies \mathrm{p}^2 = \frac{ayq^2}{x}$$

$$\therefore \; p = \sqrt{\frac{ay}{x}}q$$

Solution of Problem 6 Continue...

$$\text{Using } p=\sqrt{\frac{ay}{x}}q \text{ in equation (27)}$$

$$\frac{ayq^2}{x}x+q^2y=z$$

$$q^2y(1+a)=z \implies \mathrm{q}^2=\frac{z}{y(1+a)}$$

$$\therefore\ q=\sqrt{\frac{z}{y(1+a)}}$$

using value of q in p

$$p=\sqrt{\frac{ay}{x}}\sqrt{\frac{z}{y(1+a)}}$$

$$\therefore\ p=\sqrt{\frac{az}{x(1+a)}}$$

Solution of Problem 6 Continue...

Consider, $p\ dx + q\ dy = dz$

$$\sqrt{\frac{az}{x(1+a)}}dx + \sqrt{\frac{z}{y(1+a)}}dy = dz$$

$$\sqrt{\frac{a}{(1+a)}}\frac{dx}{\sqrt{x}} + \sqrt{\frac{1}{(1+a)}}\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Integrating we get,

$$\sqrt{\frac{a}{(1+a)}}2\sqrt{x} + \sqrt{\frac{1}{(1+a)}}2\sqrt{y} = 2\sqrt{z} + b$$

$$\sqrt{ax} + \sqrt{y} = \sqrt{a+1}(\sqrt{z} + b)$$

This is Required Complete Integral.

Problem 7:

Solve the PDE $p^2x + qy = z$ by Charpit's method

Solution

Let

$$f = p^2x + qy - z = 0 \tag{28}$$

$$\frac{\partial f}{\partial x} = f_x = p^2$$
$$\frac{\partial f}{\partial y} = f_y = q$$

Solution of Problem 6 Continue...

$$\frac{\partial f}{\partial z} = f_z = -1$$
$$\frac{\partial f}{\partial p} = f_p = 2px$$
$$\frac{\partial f}{\partial q} = f_q = y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$
$$\frac{dx}{2px} = \frac{dy}{y} = \frac{dz}{2p^2x + qy} = \frac{-dp}{p^2 - p} = \frac{-dq}{q - q}$$

Solution of Problem 7 Continue...

Consider,

$$EachRatio = \frac{dq}{0}$$

$$\therefore \ \mathrm{dq} = 0$$

Integrating,we get

$$\therefore \ \ q = a$$

Using $q = a$ in equation (28)

$$\mathrm{p}^2x + ay - z = 0 \implies \mathrm{p}^2x = z - ay \implies \mathrm{p}^2 = \frac{z - ay}{x}$$

$$p = \sqrt{\frac{z - ay}{x}}$$

$$\therefore \ \ p = \sqrt{\frac{z - ay}{x}}$$

Solution of PDE by using Charpit's Method

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Solution of Problem 7 Continue...

Consider, $p \; dx + q \; dy = dz$

$$\sqrt{\frac{z-ay}{x}}dx + ady = dz$$

$$\sqrt{z-ay}\frac{dx}{\sqrt{x}} = dz - ady$$

$$\frac{dx}{\sqrt{x}} = \frac{dz-ady}{\sqrt{z-ay}}$$

Integrating we get,

$$2\sqrt{x} = 2\sqrt{z-ay} + b$$

$$\sqrt{x} - \sqrt{z-ay} = b$$

This is Required Complete Integral.

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Type 1: PDE involving p and q only

PDE involving p and q only

The equation containing p and q only is of the form

$$f(\ p\ ,\ q\ ) = 0 \tag{29}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$
$$\frac{\partial f}{\partial y} = f_y = 0$$

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Type 1: PDE involving p and q only Continue...

$$\frac{\partial f}{\partial z} = f_z = 0$$

$$\frac{\partial f}{\partial p} = f_p$$

$$\frac{\partial f}{\partial q} = f_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{0} = \frac{-dq}{0}$$

Special types of First Order Partial Differential Equations

Type 1: PDE involving p and q only Continue...

$$\text{Now, } EachRatio = \frac{-dp}{0}$$

$$\therefore \text{dp} = 0$$

$$\text{Integrating we get ,} \quad p = a$$

Using $p = a$ in equation (29), we obtain

$$q = \Phi(a)$$

using values of p and q in $p \ dx + q \ dy = dz$

$$a \ dx + \Phi(a) \ dy = dz$$

Integrating

$$ax + \Phi(a)y = z + b$$

$$ax + \Phi(a)y - z = b$$

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Problem 1:

Solve the PDE $p + q = pq$

Solution

Let

$$f = p + q - pq = 0 \tag{30}$$

It contains only p and q.

$\therefore$ p = a = constant

$\therefore$ $p = a = constant$

Using $p = a$ in equation (30)

$\therefore$ a + q = aq

Solution of Problem 1 Continue...

$$\therefore aq - q = a$$

$$\therefore q(a - 1) = a$$

$$\therefore q = \frac{a}{a-1}$$

$$q = \frac{a}{a-1}$$

Consider, $p \; dx + q \; dy = dz$

$$a \; dx + \frac{a}{a-1} \; dy = dz$$

Integrating we get, $ax + \dfrac{ay}{a-1} = z$

$a(a-1)x + ay = (a-1)z$... Required Solution

Type 2: PDE not involving independent variables x and y

Type 2:PDE not involving independent variables x and y

The equation containing p and q only is of the form

$$f(\ p\ ,\ q\ ,\ z\ ) = 0 \tag{31}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$

$$\frac{\partial f}{\partial y} = f_y = 0$$

Type 2: PDE not involving independent variables x and y

$$\frac{\partial f}{\partial z} = f_z$$

$$\frac{\partial f}{\partial p} = f_p$$

$$\frac{\partial f}{\partial q} = f_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{pf_z} = \frac{-dq}{qf_z}$$

Type 2: PDE not involving independent variables x and y Continue

$$\implies \frac{-dp}{pf_z} = \frac{-dq}{qf_z}$$

$$\implies \frac{-dp}{p} = \frac{-dq}{q}$$

Integrating we get,

$$\ln p = \ln q + \ln a$$

$$\ln p = \ln aq$$

$$p = aq$$

Using $p = a$ in equation (31), we obtain expression for q using values of p and q in $p \ dx + q \ dy = dz$

Integrating we get required solution

Type 2: PDE not involving independent variables

Problem 1:

Solve the PDE $p^2z^2 + q^2 = 1$

Solution

It is of the form

$$f(p, q, z) = p^2z^2 + q^2 - 1 = 0 \tag{32}$$

$$\text{Put } p = aq \text{ in (32) then } a^2q^2z^2 + q^2 = 1$$

$$\implies q^2(a^2z^2 + 1) = 1 \implies q^2 = \frac{1}{a^2z^2 + 1}$$

$$q = \frac{1}{\sqrt{a^2z^2 + 1}}$$

Solution of Problem 1 Continue...

Using value of q in $p = aq$

$$p = \frac{a}{\sqrt{a^2z^2+1}}$$

Consider, $p\ dx + q\ dy = dz$

$$\frac{a}{\sqrt{a^2z^2+1}}\ dx + \frac{1}{\sqrt{a^2z^2+1}}\ dy = dz$$

$$adx + dy = \sqrt{a^2z^2+1}dz$$

Integrating we get,

$$ax + y = a\int\sqrt{z^2+\frac{1}{a^2}}$$

$$ax + y = a\left[\frac{z}{z}\sqrt{z^2+\frac{1}{a^2}} + \frac{1}{2a^2}\ln\left(z + +\sqrt{z^2+\frac{1}{a^2}}\right)\right]$$

This is Required Solution

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Problem 2:

Solve the PDE $pq + q^3 = 3pzq$

Solution

It is of the form

$$f(p, q, z) = pq + q^3 - 3pzq = 0 \tag{33}$$

$$\text{Put } p = aq \text{ in (33) then } aq^2 + q^3 = 3aq^2z$$
$$\implies q^2(a + q) = 3aq^2z$$
$$\implies (a + q) = 3az$$
$$q = 3az - a = a(3z - 1)$$

Type 2: PDE not involving independent variables

Solution of Problem 2 Continue...

Using value of q in $p = aq$

$$p = a^2(3z - 1)$$

Consider, $p\ dx + q\ dy = dz$

$$a^2(3z-1)\ dx + a(3z-1)\ dy = dz$$

$$a^2dx + ady = \frac{dz}{3z-1}$$

Integrating we get,

$$a^2x + ay = \frac{\ln(3z-1)}{3} + b$$

$$a^2x + ay = \frac{\ln(3z-1)}{3} + b$$

This is Required Solution

Type 2: PDE not involving independent variables

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Problem 3:

Find the complete integral of $z^2(p^2z^2 + q^2) = 1$

Solution

It is of the form

$$f(p,\ q,\ z) = z^2(p^2z^2 + q^2) - 1 = 0 \tag{34}$$

Put $p = aq$ in (34) then $z^2(a^2q^2z^2 + q^2) = 1$

$$\implies z^2q^2(a^2z^2 + 1) = 1 \implies q^2 = \frac{1}{z^2(a^2z^2 + 1)}$$

$$q = \frac{1}{z\sqrt{a^2z^2 + 1}}$$

Solution of Problem 3 Continue...

Using value of q in $p = aq$

$$p = \frac{a}{z\sqrt{a^2z^2+1}}$$

Consider, $p \; dx + q \; dy = dz$

$$\frac{a}{z\sqrt{a^2z^2+1}} \; dx + \frac{1}{z\sqrt{a^2z^2+1}} \; dy = dz$$

$$adx + dy = z\sqrt{1+a^2z^2}dz$$

Integrating we get,

$$ax + y = \int z\sqrt{1+a^2z^2}dz + b$$

Solution of Problem 3 Continue...

For integration, we use substitution method

$$\text{Put } 1 + a^2z^2 = t$$

$$\implies 2a^2zdz = dt$$

$$\implies z\ dz = \frac{dt}{2a^2}$$

$$ax + y - b = \int \sqrt{t}\frac{dt}{2a^2}$$

$$ax + y - b = \frac{1}{2a^2}\frac{t^{\frac{3}{2}}}{\frac{3}{2}}$$

$$3a^2(ax + y - b) = (a^2z^2 + 1)^{\frac{3}{2}}$$

This is Required Solution.

Type 2: PDE not involving independent variables

Problem 4:

Find the complete integral of $q^2 = z^2p^2(1 - p^2)$

Solution

It is of the form

$$\text{f(p, q, z)} = \text{q}^2 - z^2p^2(1 - p^2) = 0 \tag{35}$$

Put $p = aq$ in (34) then $q^2 = z^2a^2q^2(1 - a^2q^2)$

$$\implies 1 - \text{a}^2q^2 = \frac{1}{a^2z^2} \implies 1 - \frac{1}{a^2z^2} = a^2q^2$$

$$\implies \frac{a^2z^2 - 1}{a^2z^2} = a^2q^2$$

Solution of Problem 4 Continue...

$$\therefore \quad q^2 = \frac{a^2z^2 - 1}{a^4z^2}$$

$$q = \frac{\sqrt{z^2a^2 - 1}}{a^2z}$$

Using value of q in $p = aq$

$$p = \frac{a\sqrt{z^2a^2 - 1}}{a^2z}$$

$$p = \frac{\sqrt{z^2a^2 - 1}}{az}$$

Consider, $p \; dx + q \; dy = dz$

$$\frac{\sqrt{z^2a^2 - 1}}{az} \; dx + \frac{\sqrt{z^2a^2 - 1}}{a^2z} \; dy = dz$$

$$adx + dy = \frac{a^2z}{\sqrt{z^2a^2 - 1}}dz$$

Type 2: PDE not involving independent variables

Solution of Problem 4 Continue...

Integrating we get,

$$ax + y = \int \frac{a^2 z}{\sqrt{z^2 a^2 - 1}} dz + b$$

For integration, we use substitution method

Put $a^2 z^2 - 1 = t$

$$\implies 2a^2 z dz = dt \implies a^2 z dz = \frac{dt}{2}$$

$$ax + y = \int \sqrt{t} \frac{dt}{2\sqrt{t}} + b$$

$$ax + y = \frac{2\sqrt{t}}{2}$$

$$ax + y = \sqrt{a^2 z^2 - 1} + b$$

This is Required Solution.

Type 3: Separable Form

Type 3: Separable Form

The partial differential equation is said to be separable if it can be written in the form

$$f(\ x\ ,\ p\ ) = g(\ y\ ,\ q\ ) \tag{36}$$

$$\text{Let } F = f(x,p) - g(y,q) = 0$$
$$\frac{\partial F}{\partial x} = F_x = f_x$$
$$\frac{\partial F}{\partial y} = F_y = -g_x$$

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Type 3: Separable Form continue...

$$\frac{\partial F}{\partial z} = F_z = O$$

$$\frac{\partial F}{\partial p} = F_p = f_p$$

$$\frac{\partial F}{\partial q} = F_q = -g_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{p\,f_p - q\,g_q} = \frac{-dp}{f_x} = \frac{-dq}{-g_y}$$

Special types of First Order Partial Differential Equations

Type 3: Separable Form Continue ...

Consider, $\dfrac{dx}{f_p} = \dfrac{-dp}{f_x}$

$\therefore \mathrm{f}_x dx + f_p dp = 0$

$\therefore \mathrm{d[f(x\ ,\ p)]} = 0$

Integrating we get,

$f(x, p) = a$

Similarly using this in (36)

$g(y, q) = a$

obtain expression for p and q from above two equations

using values of p and q in $p\ dx + q\ dy = dz$

Integrating we get required solution.

Type 3: Separable Form

Problem 1:

Find the complete integral of $p^2 + q^2 = x + y$

Solution

Given equation is

$$p^2 + q^2 = x + y$$

$$\therefore \mathrm{p}^2 - x = y - q^2$$

It is of the form

$$f(x, p) = g(y, q)$$

$$\text{Let } f(x, p) = p^2 - x = a$$

$$\therefore \mathrm{p}^2 = a + x$$

$$p = \sqrt{x + a}$$

Solution of Problem 1 Continue...

Similarly, $g(y,q) = y - q^2 = a$

$\therefore \text{q}^2 = y - a$

$q = \sqrt{y-a}$

Consider, $p\ \ dx + q\ \ dy = dz$

$$\sqrt{x+a}\ \ dx + \sqrt{y-a}\ \ dy = dz$$

$$(x+a)^{\frac{1}{2}}dx + (y-a)^{\frac{1}{2}}dy = dz$$

Integrating we get,

$$\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(y-a)^{\frac{3}{2}}}{\frac{3}{2}} = z + b$$

$$(x+a)^{\frac{3}{2}} + (y-a)^{\frac{3}{2}} = \frac{3}{2}(z+b)$$

This is required solution.

Type 3: Separable Form

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Problem 2:

Find the complete integral of $p^2y(1+x^2) = qx^2$

Solution

Given equation is

$$p^2y(1+x^2) = qx^2$$

$$\therefore \ p^2\left(\frac{1+x^2}{x^2}\right) = \frac{q}{y}$$

It is of the form

$$f(x,p) = g(y,q)$$

$$\text{Let } f(x,p) = p^2\left(\frac{1+x^2}{x^2}\right) = a$$

Type 3: Separable Form

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Solution of Problem 2 Continue...

$$\therefore \; \mathrm{p}^2 = a\left(\frac{x^2}{1+x^2}\right)$$

$$p = x\sqrt{\frac{a}{1+x^2}}$$

$$\text{Similarly, } g(y,q) = \frac{q}{y} = a$$

$$q = ay$$

$$\text{Consider, } p\;dx + q\;dy = dz$$

$$\sqrt{\frac{a}{1+x^2}}\;xdx + ay\;dy = dz$$

$$\sqrt{a}\frac{xdx}{\sqrt{1+x^2}} + aydy = dz$$

Type 3: Separable Form

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Solution of Problem 2 Continue...

Integrating we get,

$$\sqrt{a}\sqrt{1+x^2} + a\frac{y^2}{2} = z + b$$

$$2\sqrt{a}\sqrt{1+x^2} + ay^2 = 2z + b$$

This is required solution.

Special types of First Order Partial Differential Equations

Type 4: Claraut's Equation:

Type 4: Claraut's Equation:

The partial differential equation is of the form $z = px + qy + f(p, q)$ Where x and y are independent and z is dependent variable called as Claraut's Equation.

$$\text{Now let } F = px + qy + f(x, p) - z = 0$$

$$\frac{\partial F}{\partial x} = F_x = p$$

$$\frac{\partial F}{\partial y} = F_y = q$$

Type 4: Claraut's Equation continue...

$$\frac{\partial F}{\partial z} = F_z = -1$$

$$\frac{\partial F}{\partial p} = F_p = x + f_p$$

$$\frac{\partial F}{\partial q} = F_q = y + f_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{dz}{xp + pf_p + yq + qf_q} = \frac{-dp}{p - p} = \frac{-dq}{q - q}$$

Type 4: Claraut's Equation continue...

$$\text{Consider, Each Ratio} = \frac{dp}{0}$$

$$\therefore \text{dp} = 0$$

Integrating we get,

$$p = a$$

$$\text{Now Consider, Each Ratio} = \frac{dq}{0}$$

$$\therefore \text{dq} = 0$$

Integrating we get,

$$q = b$$

using values of p and q in $z = px + qy + f(p, q)$

$$z = ax + by + f(a, b)$$

Integrating we get required solution.

Type 4: Claraut's Equation:

Problem 1:

Find the complete integral of $(p+q)(z-px-qy)=1$

Solution

Given equation is

$$z - px - qy = \frac{1}{p+q}$$

$$z = px + qy + \frac{1}{p+q}$$

It is of the Claraut's Equation form

Hence its solution is,

$$z = ax + by + \frac{1}{a+b}$$

where a and b are constants.

Jacobi's Auxiliary Equation

Show that the Jacobi's Auxiliary Equation is

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

Proof

This method is used for solving first order partial differential equation involving 3 or more independent variables.

Let

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \tag{37}$$

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

be first order partial differential equation where z is function of x_1, x_2 and x_3 and,

$$p_1 = \frac{\partial z}{\partial x_1}, \quad p_2 = \frac{\partial z}{\partial x_2}, \quad p_3 = \frac{\partial z}{\partial x_3}$$

such that

$$\mathrm{p}_1 \; dx_1 + p_2 \; dx_2 + p_3 \; dx_3 = dz \tag{38}$$

The Jacobi's method is same is same as that of Charpit's method. The main thing of Jacobi's method is to obtain two additional equations.

Proof of Jacobi's Auxiliary Equation...

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \tag{39}$$

and

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2 \tag{40}$$

where a_1 and a_2 are arbitrary constants.
We find p_1, p_2, p_3 from (37) , (39) and (40) such that equation (38) will be integrable.

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Proof of Jacobi's Auxiliary Equation...

Differentiate (37) and (39) w.r.t. x_1

$$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial p_1}.\frac{\partial p_1}{\partial x_1}+\frac{\partial f}{\partial p_2}.\frac{\partial p_2}{\partial x_1}+\frac{\partial f}{\partial p_3}.\frac{\partial p_3}{\partial x_1}=0 \qquad (41)$$

Similarly,

$$\frac{\partial F_1}{\partial x_1}+\frac{\partial F_1}{\partial p_1}.\frac{\partial p_1}{\partial x_1}+\frac{\partial F_1}{\partial p_2}.\frac{\partial p_2}{\partial x_1}+\frac{\partial F_1}{\partial p_3}.\frac{\partial p_3}{\partial x_1}=0 \qquad (42)$$

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

Multiply (41) by $\dfrac{\partial F_1}{\partial p_1}$ and (42) by $\dfrac{\partial f}{\partial p_1}$ then take (41) - (42)

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial x_1}\frac{\partial f}{\partial p_1}\right) + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial p_2}\frac{\partial f}{\partial p_1}\right)\frac{\partial p_2}{\partial x_1} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial p_3}\frac{\partial f}{\partial p_1}\right)\frac{\partial p_3}{\partial x_1} = 0 \tag{43}$$

Proof of Jacobi's Auxiliary Equation...

Similarly, Differentiate (37) and (39) w.r.t. x_2
Replace $x_1 = x_2, p_1 = p_2, p_3 = constant$

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_1}{\partial x_2}\frac{\partial f}{\partial p_2}\right) + \left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_1}{\partial p_1}\frac{\partial f}{\partial p_2}\right)\frac{\partial p_1}{\partial x_2} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_1}{\partial p_3}\frac{\partial f}{\partial p_2}\right)\frac{\partial p_3}{\partial x_2} = 0 \tag{44}$$

Proof of Jacobi's Auxiliary Equation...

Similarly, Differentiate (37) and (39) w.r.t. x_3
Replace $x_1 = x_3, p_1 = p_3, p_2 = constant$

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3} - \frac{\partial F_1}{\partial x_3}\frac{\partial f}{\partial p_3}\right) + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3} - \frac{\partial F_1}{\partial p_2}\frac{\partial f}{\partial p_3}\right)\frac{\partial p_2}{\partial x_3} + \left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3} - \frac{\partial F_1}{\partial p_1}\frac{\partial f}{\partial p_3}\right)\frac{\partial p_1}{\partial x_3} = 0 \tag{45}$$

Adding (43) , (44) and (45) and using

$$\frac{\partial p_2}{\partial x_1} = \frac{\partial}{\partial x_1}\frac{\partial z}{\partial x_2} = \frac{\partial^2 z}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2}\frac{\partial z}{\partial x_1} = \frac{\partial p_1}{\partial x_2}$$

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Proof of Jacobi's Auxiliary Equation...

using $\dfrac{\partial p_2}{\partial x_1} = \dfrac{\partial p_1}{\partial x_2}$, $\dfrac{\partial p_3}{\partial x_1} = \dfrac{\partial p_1}{\partial x_3}$ and $\dfrac{\partial p_3}{\partial x_2} = \dfrac{\partial p_2}{\partial x_3}$ we get,

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial x_1}\frac{\partial f}{\partial p_1}\right) + \left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_1}{\partial x_2}\frac{\partial f}{\partial p_2}\right) + \left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3} - \frac{\partial F_1}{\partial x_3}\frac{\partial f}{\partial p_3}\right) = 0$$

$$\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_1}{\partial p_r} - \frac{\partial F_1}{\partial x_r}\frac{\partial f}{\partial p_r}\right) = 0$$

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

It's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

This relation is known as Jacobi's Auxiliary Equation. Similarly from (37) and (40) we get,

$$\sum_{r=1}^{3} \left(\frac{\partial f}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_2}{\partial x_r} \frac{\partial f}{\partial p_r} \right) = 0$$

After finding F-1= $a_1 and F_2 = a_2$.solving the equations

Problem 1:

Solve the PDE ${p_1}^3 + {p_2}^2 + p_3 = 1$ by Jacobi's method

Solution

Let

$$f = \mathrm{p}_1^3 + {p_2}^2 + p_3 - 1 = 0 \tag{46}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = 0$$
$$\frac{\partial f}{\partial x_2} = f_{x_2} = 0$$

Solution of Problem 1 Continue...

$$\frac{\partial f}{\partial x_3} = f_{x_3} = 0$$

$$\frac{\partial f}{\partial p_1} = f_{p_1} = 3{p_1}^2$$

$$\frac{\partial f}{\partial p_2} = f_{p_2} = 2p_2$$

$$\frac{\partial f}{\partial p_3} = f_{p_3} = -1$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

Solution of Problem 1 Continue...

$$\frac{dx_1}{-3{p_1}^2} = \frac{dx_2}{2p_2} = \frac{dx_3}{-1} = \frac{dp_1}{0} = \frac{dp_2}{0} = \frac{dp_3}{0}$$

$$EachRatio = \frac{dp_1}{0} \implies \mathrm{dp}_1 = 0$$

Integrating,

$p_1 = a$.....where a is constant

Similarly, $EachRatio = \frac{dp_2}{0} \implies \mathrm{dp}_2 = 0$

Integrating,

$p_2 = b$.....where b is constant

Using $p_1 = a$ and $p_2 = b$ in equation (46)

$$a^3 + b^2 + p_3 = 1 \implies p_3 = 1 - a^3 - b^2$$

Solution of Problem 1 Continue...

Using the values of p_1, p_2 and p_3 in

$$p_1 \ dx_1 + p_2 \ dx_2 + p_3 \ dx_3 = dz$$

Consider, $p_1 \ dx_1 + p_2 \ dx_2 + p_3 \ dx_3 = dz$

$$\therefore \ \mathrm{a} \quad \mathrm{dx}_1 + b \ dx_2 + (1 - a^3 - b^2) \ dx_3 = dz$$

Integrating we get,

$$a \ x_1 + b \ x_2 + (1 - a^3 - b^2) \ x_3 = z + c$$

Required Solution.

Solution of PDE by using Jacobi's Method

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Problem 2:

Solve the PDE $2\,p_1\,x_1\,x_3 + 3\,p_2\,{x_3}^2 + {p_2}^2\,p_3 = 0$ by Jacobi's method

Solution

Let

$$f = 2\,p_1\,x_1\,x_3 + 3\,p_2\,{x_3}^2 + {p_2}^2\,p_3 = 0 \tag{47}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = 2\,p_1\,x_3$$

$$\frac{\partial f}{\partial x_2} = f_{x_2} = 0$$

Solution of Problem 2 Continue...

$$\frac{\partial f}{\partial x_3} = f_{x_3} = 2\,p_1\,x_1 + 6\,p_2\,x_3$$

$$\frac{\partial f}{\partial p_1} = f_{p_1} = 2\,x_1\,x_3$$

$$\frac{\partial f}{\partial p_2} = f_{p_2} = 3\,{x_3}^2 + 2\,p_2\,p_3$$

$$\frac{\partial f}{\partial p_3} = f_{p_3} = {p_2}^2$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

Solution of Problem 2 Continue...

$$\frac{dx_1}{-(2\ x_1\ x_3)} = \frac{dx_2}{-(3\ {x_3}^2 + 2\ p_2\ p_3)} = \frac{dx_3}{-{p_2}^2} = \frac{dp_1}{2\ p_1\ x_3} = \frac{dp_2}{0} = \frac{dp_3}{2\ p_1\ x_1 + 6\ p_2\ x_3}$$

$$EachRatio = \frac{dp_2}{0} \implies \mathrm{dp}_2 = 0$$

Integrating,

$p_2 = a$.....where a is constant

Similarly, Consider,

$$\frac{dx_1}{-(2\,x_1\,x_3)} = \frac{dp_1}{2\,p_1\,x_3}$$

$$\frac{dx_1}{-x_1} = \frac{dp_1}{p_1}$$

Integrating,

$$-\ln x_1 = \ln p_1 - \ln b$$

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Solution of Problem 2 Continue...

$\ln x_1 + \ln p_1 = \ln b$

$\ln x_1 p_1 = \ln b \implies x_1 p_1 = b$

$p_1 = \dfrac{b}{x_1}$.....where b is constant

Using $p_1 = \dfrac{b}{x_1}$ and $p_2 = a$ in equation (47)

$2\,\dfrac{b}{x_1}\,x_1\,x_3 + 3\,a\,{x_3}^2 + a^2\,p_3 = 0$

$a^2\,p_3 = -2\,b\,x_3 - 3\,a\,{x_3}^2$

$p_3 = -\dfrac{1}{a^2}(2\,b\,x_3 + 3\,a\,{x_3}^2)$

Solution of Problem 2 Continue...

Using the values of p_1, p_2 and p_3 in

$p_1 \ dx_1 + p_2 \ dx_2 + p_3 \ dx_3 = dz$

Consider,

$$p_1 \ dx_1 + p_2 \ dx_2 + p_3 \ dx_3 = dz$$

$$\therefore \quad \frac{b}{x_1} \ dx_1 + a \ dx_2 - \frac{1}{a^2}(2\,b\,x_3 + 3\,a\,{x_3}^2)dx_3 = dz$$

Integrating we get,

$$b \ln x_1 + a\,x_2 - \frac{1}{a^2}\left(2\,b\frac{{x_3}^2}{2} + 3\,a\frac{{x_3}^3}{3}\right) = z + c$$

$$b \ln x_1 + a\,x_2 - \frac{b}{a^2}\,{x_3}^2 - \frac{{x_3}^3}{a} = z + c$$

Required Solution.

Solution of PDE by using Jacobi's Method

Problem 3:

Solve the PDE $p_1\,x_1 + p_2\,x_2 = {p_3}^2$ by Jacobi's method

Solution

Let

$$f = p_1\,x_1 + p_2\,x_2 - {p_3}^2 = 0 \tag{48}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = p_1$$

$$\frac{\partial f}{\partial x_2} = f_{x_2} = p_2$$

Solution of Problem 3 Continue...

$$\frac{\partial f}{\partial x_3} = f_{x_3} = 0$$

$$\frac{\partial f}{\partial p_1} = f_{p_1} = x_1$$

$$\frac{\partial f}{\partial p_2} = f_{p_2} = x_2$$

$$\frac{\partial f}{\partial p_3} = f_{p_3} = -2p_3$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

Solution of Problem 3 Continue...

$$\frac{dx_1}{-x_1} = \frac{dx_2}{-x_2} = \frac{dx_3}{-(-2p_3)} = \frac{dp_1}{p_1} = \frac{dp_2}{p_2} = \frac{dp_3}{0}$$

$$EachRatio = \frac{dp_3}{0} \implies \mathrm{dp}_3 = 0$$

Integrating,

$p_3 = a$.....where a is constant

Similarly, Consider,

$$\frac{dx_1}{-x_1} = \frac{dp_1}{p_1}$$

Integrating,

$$-\ln x_1 = \ln p_1 - \ln b$$

Solution of Problem 3 Continue...

$\ln x_1 + \ln p_1 = \ln b$

$\ln x_1 p_1 = \ln b \implies x_1 p_1 = b$

$p_1 = \dfrac{b}{x_1}$.....where b is constant

Using $p_1 = \dfrac{b}{x_1}$ and $p_3 = a$ in equation (48)

$2\,\dfrac{b}{x_1}\,x_1 + p_2\,x_2 - a^2 = 0$

$p_2\,x_2 = a^2 - b$

$p_2 = \dfrac{a^2 - b}{x_2}$

Solution of PDE by using Jacobi's Method

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Solution of Problem 3 Continue...

Using the values of p_1, p_2 and p_3 in

$p_1 \ dx_1 + p_2 \ dx_2 + p_3 \ dx_3 = dz$

Consider,

$p_1 \ dx_1 + p_2 \ dx_2 + p_3 \ dx_3 = dz$

$$\therefore \quad \frac{b}{x_1} \ dx_1 + \frac{a^2 - b}{x_2} \ dx_2 + a dx_3 = dz$$

Integrating we get,

$b \ln x_1 + (a^2 - b) \ln x_2 + a \, x_3 = z + c$

$b \ln x_1 + (a^2 - b) \ln x_2 + a \, x_3 = z + c$

Required Solution.

Problem 4:

Solve the PDE $p_1\, p_2\, p_3 = z^3\, x_1\, x_2\, x_3$ by Jacobi's method

Solution

Let

$$p_1\, p_2\, p_3 = z^3\, x_1\, x_2\, x_3$$

Dividing by z^3

$$\left(\frac{1}{z}\, p_1\right)\left(\frac{1}{z}\, p_2\right)\left(\frac{1}{z}\, p_3\right) = x_1\, x_2\, x_3$$

Put $u = \log z$

Differentiate w.r.t. x_1, we get

$$\therefore \quad \frac{\partial u}{\partial x_1} = \frac{1}{z}\frac{\partial z}{\partial x_1} = \frac{1}{z}\, p_1$$

Solution of PDE by using Jacobi's Method

Solution of Problem 4 Continue...

$$\text{Take } P_1 = \frac{1}{z}\,p_1, \quad P_2 = \frac{1}{z}\,p_2, \quad P_3 = \frac{1}{z}\,p_3$$

Equation becomes,

$$P_1\,P_2\,P_3 = x_1\,x_2\,x_3$$

$$\mathrm{f} = \mathrm{P}_1\,P_2\,P_3 - x_1\,x_2\,x_3 = 0 \tag{49}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = -x_2\,x_3,$$
$$\frac{\partial f}{\partial x_2} = f_{x_2} = -x_1\,x_3,$$
$$\frac{\partial f}{\partial x_3} = f_{x_3} = -x_1\,x_2;$$

Solution of Problem 4 Continue...

$$\frac{\partial f}{\partial P_1} = f_{P_1} = P_2 P_3$$
$$\frac{\partial f}{\partial P_2} = f_{P_2} = P_1 P_3$$
$$\frac{\partial f}{\partial P_3} = f_{P_3} = P_1 P_2$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{P_1}} = \frac{dx_2}{-f_{P_2}} = \frac{dx_3}{-f_{P_3}} = \frac{dP_1}{f_{x_1}} = \frac{dP_2}{f_{x_2}} = \frac{dP_3}{f_{x_3}}$$

$$\frac{dx_1}{-P_2 P_3} = \frac{dx_2}{-P_1 P_3} = \frac{dx_3}{-P_1 P_2} = \frac{dP_1}{-x_2 x_3} = \frac{dP_2}{-x_1 x_3} = \frac{dP_3}{-x_1 x_2}$$

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Solution of Problem 4 Continue...

Consider

$$\frac{dx_1}{-P_2P_3} = \frac{dP_1}{-x_2x_3}$$

$$\therefore \quad \frac{P_1\,dx_1}{P_1P_2P_3} = \frac{dP_1}{x_2x_3}$$

$$\therefore \quad \frac{P_1\,dx_1}{x_1x_2x_3} = \frac{dP_1}{x_2x_3} \quad \text{..... By Equation (49)}$$

$$\therefore \quad \frac{dx_1}{x_1} = \frac{dP_1}{P_1}$$

Integrating,

$$\therefore \quad \log x_1 = \log P_1 - \log a$$

$$\therefore \quad \log x_1 + \log a = \log P_1$$

$$P_1 = ax_1 \text{.....where a is constant}$$

Solution of Problem 4 Continue...

Similarly, Consider,

$$\frac{dx_2}{-P_1P_3} = \frac{dP_2}{-x_1x_3}$$

$$\therefore \quad \frac{P_2dx_2}{P_1P_2P_3} = \frac{dP_2}{x_1x_3}$$

$$\therefore \quad \frac{P_2dx_2}{x_1x_2x_3} = \frac{dP_2}{x_1x_3} \text{... by equation (49)}$$

$$\therefore \quad \frac{dx_2}{x_2} = \frac{dP_2}{P_2}$$

Integrating,

$$\therefore \quad \log x_2 = \log P_2 - \log b$$

$$\therefore \quad \log x_2 + \log b = \log P_2$$

$P_2 = bx_2$.....where b is constant

Solution of Problem 4 Continue...

Using $P_1 = ax_1$ and $P_2 = bx_2$ in equation (49)

$$\therefore \text{a} \quad \text{x}_1\, b\, x_2\, P_3 - x_1 x_2 x_3 = 0$$

$$\therefore \text{a} \quad \text{x}_1\, b\, x_2\, P_3 = x_1 x_2 x_3$$

$$\therefore \text{P}_3 = \frac{x_3}{ab}$$

$$P_3 = \frac{x_3}{ab}$$

Using the values of p_1, p_2 and p_3 in in

$$P_1 \;\; dx_1 + P_2 \;\; dx_2 + P_3 \;\; dx_3 = dz$$

$$\therefore \;\; \text{a}\; \text{x}_1 \;\; dx_1 + bx_2 \;\; dx_2 + \frac{x_3}{ab} dx_3 = dz$$

Integrating we get,

$$\frac{ax_1{}^2}{2} + \frac{bx_2{}^2}{2} + \frac{x_3{}^2}{2ab} z + c$$

$a^2 bx_1{}^2 + ab^2 x_2{}^2 + x_3{}^2 = 2abz + c$... Required Solution.

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Problem 5:

Solve the PDE $p^2 x + q^2 y = z$ by Jacobi's method

Solution

Jacobi's method is used for solving first order partial differential equation involving 3 or more independent variables. Here x and y are independent and z is dependent variable. So we consider z as independent variable

if and only if $u(x, y, z) = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \implies \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p = 0$$

Solution of PDE by using Jacobi's Method

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Solution of Problem 4 Continue...

Let

$$u_1 = \frac{\partial u}{\partial x}, \quad u_2 = \frac{\partial u}{\partial y}, \quad u_3 = \frac{\partial u}{\partial z}$$

$$\therefore \ \mathrm{u}_1 + u_3 \ p = 0$$

$$\therefore \ \ \mathrm{p} = \text{-} \ \frac{u_1}{u_3}$$

$$\text{Similarly, } q = -\frac{u_2}{u_3}$$

Using this in (1)

$$\frac{{u_1}^2}{{u_3}^2} x + \frac{{u_2}^2}{{u_3}^2} y = z$$

$$\therefore \ \ \mathrm{u}_1^2 x + {u_2}^2 y = {u_3}^2 z$$

Solution of Problem 4 Continue...

Let

$$f = u_1^2 x + {u_2}^2 y - {u_3}^2 z = 0 \tag{50}$$

$$\frac{\partial f}{\partial x} = f_x = {u_1}^2,$$
$$\frac{\partial f}{\partial y} = f_y = {u_2}^2,$$
$$\frac{\partial f}{\partial z} = f_z = -{u_3}^2;$$
$$\frac{\partial f}{\partial u_1} = f_{u_1} = 2u_1 x$$
$$\frac{\partial f}{\partial u_2} = f_{u_2} = 2u_2 y \text{ and } \frac{\partial f}{\partial u_3} = f_{u_3} = -2u_3 z$$

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Solution of Problem 4 Continue...

Jacobi's Auxiliary equation is,

$$\frac{dx}{-f_{u_1}} = \frac{dy}{-f_{u_2}} = \frac{dz}{-f_{u_3}} = \frac{du_1}{f_x} = \frac{du_2}{f_y} = \frac{du_3}{f_z}$$

$$\frac{dx}{-2u_1\,x} = \frac{dy}{-2u_2\,y} = \frac{dz}{-2u_3\,z} = \frac{du_1}{{u_1}^2} = \frac{du_2}{{u_2}^2} = \frac{du_3}{-{u_3}^2}$$

Consider

$$\frac{dx}{-2u_1\,x} = \frac{du_1}{{u_1}^2}$$

$$\frac{dx}{-2\,x} = \frac{du_1}{u_1}$$

Solution of Problem 4 Continue...

Integrating,

$$\therefore \quad -\frac{1}{2}\log x = \log u_1 - \log a$$

$$\therefore \quad \log {u_1}^2 + \log x = \log a$$

$$\therefore \quad \log {u_1}^2 x = \log a$$

$$\therefore \quad {u_1}^2 x = a$$

$$u_1 = \sqrt{\frac{a}{x}}\text{.....where a is constant}$$

Similarly, Consider,

$$\frac{dy}{-2u_2\,y} = \frac{du_2}{{u_2}^2}$$

$$\frac{dy}{-2\,y} = \frac{du_2}{u_2}$$

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Solution of Problem 4 Continue...

Integrating,

$$\therefore \quad -\frac{1}{2}\log y = \log u_2 - \log b$$

$$\therefore \quad \log {u_2}^2 + \log y = \log b$$

$$\therefore \quad \log {u_2}^2 y = \log b \therefore \quad {u_2}^2 y = b$$

$$u_2 = \sqrt{\frac{b}{y}}$$ .....where b is constant

Using $u_1 = \sqrt{\frac{a}{x}}$ and $u_2 = \sqrt{\frac{b}{y}}$ in equation (50)

$$\therefore \quad \frac{a}{x}x + \frac{b}{y}y - {u_3}^2 z = 0 \implies \therefore \; u_3^2 z = a + b$$

$$u_3 = \sqrt{\frac{a+b}{z}}$$

Solution of Problem 4 Continue...

Using the values of u_1, u_2 and u_3 in in

$$u_1 \ dx + u_2 \ dy + u_3 \ dz = du$$

$$\therefore \ \sqrt{\frac{a}{x}} \ dx + \sqrt{\frac{b}{y}} \ dy + \sqrt{\frac{a+b}{z}} \ dz = du$$

$$\therefore \ \sqrt{a} \ \frac{dx}{\sqrt{x}} + \sqrt{b} \ \frac{dy}{\sqrt{y}} + \sqrt{a+b} \ \frac{dz}{\sqrt{z}} = du$$

Integrating we get,

$$2\sqrt{ax} + 2\sqrt{by} + 2\sqrt{(a+b)z} = u + c$$

But $u(x, y, z) = 0$

$$\sqrt{ax} + \sqrt{by} + \sqrt{(a+b)z} = c$$

Required Solution.