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Tutorial No. 2: Find general solution of Langrange's equation.

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Find general solution of Langrange's equation

General Solution of Lagrange's Equation

Lagrange's Auxiliary Equation

$$P(x, y, z) \, p + Q(x, y, z) \, q = R(x, y, z)$$

where P , Q and R are continuously differentiable functions on the domain $D \subseteq R^3$ is $\phi(u, v) = 0$ where ϕ is an arbitrary function and

$$u(x, y, z) = c_1 \text{ and } v(x, y, z) = c_2$$

are two independent solutions of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

This equation is known as Lagrange's Auxiliary Equation

General Solution of Lagrange's Equation

Problem 1:

Solve the PDE of $\frac{y^2 z}{x} p + xz q = y^2$

Solution

It is of the form

$$P p + Q q = R$$

Lagrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \quad (1)$$

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Solution of Problem 1 Continue...

Taking the first two fractions from (1)

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz}$$

$$x^2 dx = y^2 dy$$

Integrating we get, $x^3 = y^3 + c_1$

$$x^3 - y^3 = c_1 \quad (2)$$

General Solution of Lagrange's Equation

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Solution of Problem 1 Continue...

Taking the first and last fractions from (1)

$$\frac{dx}{y^2 z} = \frac{dz}{y^2}$$

$$x dx = z dz$$

Integrating we get, $x^2 = z^2 + c_2$

$$x^2 - z^2 = c_2 \quad (3)$$

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Solution of Problem 1 Continue...

From equations (2) and (3), the required general solution
is

$$\Phi(x^3 - y^3, x^2 - z^2)$$

where ϕ is arbitrary function.

General Solution of Lagrange's Equation

Problem 2:

Solve the PDE of $p + 3q = 5z + \tan(y - 3x)$

Solution

It is of the form

$$Pp + Qq = R$$

Lagrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)} \quad (4)$$

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Solution of Problem 2 Continue...

Taking the first two fractions from (4)

$$\frac{dx}{1} = \frac{dy}{3}$$
$$3dx = ydy$$

Integrating we get, $3x = y + c_1$

$$3x - y = c_1 \quad (5)$$

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Solution of Problem 2 Continue...

Taking the first and last fractions from (4)

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y - 3x)}$$

$$dx = \frac{dz}{5z + \tan(c_1)} \dots \text{From equation (5)}$$

$$\text{Integrating we get, } x = \frac{\ln(5z + \tan c_1)}{5}$$

$$5x - \ln(5z + \tan c_1) = c_2 \quad (6)$$

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Solution of Problem 2 Continue...

From equations (5) and (6), the required general solution
is

$$\Phi(3x - y, 5x - \ln[5z + \tan(3x - y)])$$

where ϕ is arbitrary function.

General Solution of Langrange's Equation

Problem 3:

Solve the PDE of $yz \frac{b-c}{a} p + zx \frac{c-a}{b} q = \frac{a-b}{c} xy$

Solution

It is of the form

$$P p + Q q = R$$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{yz(b-c)} = \frac{dy}{xz(c-a)} = \frac{dz}{xy(a-b)} \quad (7)$$

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Solution of Problem 3 Continue...

Choosing x, y, z as multipliers, each fraction equals

$$\begin{aligned} &= \frac{axdx}{xyz(b-c)} + \frac{bydy}{xyz(c-a)} + \frac{czdz}{xy(a-b)} \\ &= \frac{axdx + bydy + czdz}{xyz(b-c+c-a+a-b)} \implies axdx + bydy + czdz = 0 \end{aligned}$$

Integrating we get, $\frac{ax^2}{2} + \frac{by^2}{2} + \frac{cz^2}{2} = c_1$

$$ax^2 + by^2 + cz^2 = c_1 \quad (8)$$

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Solution of Problem 3 Continue...

Choosing ax, by, cz as multipliers, each fraction equals

$$e = \frac{a^2 x dx}{xyz(b-c)} + \frac{b^2 y dy}{xyz(c-a)} + \frac{c^2 z dz}{xyz(a-b)}$$

$$= \frac{a^2 x dx + b^2 y dy + c^2 z dz}{xyz(ab - ac + bc - ba + ca - cb)}$$

$$\implies a^2 x dx + b^2 y dy + c^2 z dz = 0$$

Integrating we get, $\frac{a^2 x^2}{2} + \frac{b^2 y^2}{2} + \frac{c^2 z^2}{2} = c_2$

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Solution of Problem 3 Continue...

$$a^2x^2 + b^2y^2 + c^2z^2 = c_2 \quad (9)$$

From equations (8) and (9), the required general solution is

$$\Phi(ax^2 + by^2 + cz^2, a^2x^2 + b^2y^2 + c^2z^2)$$

where ϕ is arbitrary function.

General Solution of Lagrange's Equation

Problem 4:

Solve the PDE of $(y + z) p + (z + x) q = x + y$

Solution

It is of the form

$$P p + Q q = R$$

Lagrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y + z} = \frac{dy}{x + z} = \frac{dz}{x + y} \quad (10)$$

General Solution of Lagrange's Equation

Solution of Problem 4 Continue...

Choosing 1, -1, 0 as multipliers, each fraction equals

$$= \frac{dx - dy}{y - x} = - \frac{d(x - y)}{x - y}$$

Choosing 0, 1, -1 as multipliers, each fraction equals

$$= \frac{dy - dz}{z - y} = - \frac{d(y - z)}{y - z}$$

Choosing 1, 1, 1 as multipliers, each fraction equals

$$\begin{aligned} &= \frac{dx + dy + dz}{y + z + z + x + x + y} \\ &= \frac{dx + dy + dz}{2(x + y + z)} \end{aligned}$$

General Solution of Lagrange's Equation

Solution of Problem 4 Continue...

Taking above three ratios,

$$-\frac{d(x-y)}{x-y} = -\frac{d(y-z)}{y-z} = \frac{dx+dy+dz}{2(x+y+z)}$$

taking the first two fraction

$$-\frac{d(x-y)}{x-y} = -\frac{d(y-z)}{y-z}$$

Integrating we get,

$$\ln x - y = \ln y - z + \ln c_1$$

$$\frac{x-y}{y-z} = c_1 \quad (11)$$

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Solution of Problem 4 Continue...

Taking First and Third ratios,

$$-\frac{d(x-y)}{x-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

Integrating we get,

$$-2 \ln x - y = \ln(x+y+z) + \ln c_2$$

$$\ln(x-y)^2 + \ln(x+y+z) = -\ln c_2$$

$$\ln(x-y)^2(x+y+z) = \ln c_2$$

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Solution of Problem 4 Continue...

$$(x+y+z)(x-y)^2 = c_2 \quad (12)$$

From equations (11) and (12), the required general solution is

$$\Phi\left(\frac{x-y}{y-z}, (x+y+z)(x-y)^2\right)$$

where ϕ is arbitrary function.

Problem 5:

Solve the PDE of $(x + 2z) p + (4xz - y) q = 2x^2 + y$

Solution

It is of the form

$$P p + Q q = R$$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{(x + 2z)} = \frac{dy}{(4xz - y)} = \frac{dz}{(2x^2 + y)} \quad (13)$$

General Solution of Lagrange's Equation

Solution of Problem 5 Continue...

Choosing $-2x, 1, 1$ as multipliers, each fraction equals

$$= \frac{-2xdx + dy + dz}{-2x^2 - 4xz + 4xz - y + 2x^2 + y}$$

$$\therefore -2xdx + dy + dz = 0$$

Integrating we get,

$$\therefore \frac{-2x^2}{2} + y + z = c_1$$

$$-x^2 + y + z = c_1 \quad (14)$$

General Solution of Lagrange's Equation

Solution of Problem 5 Continue...

Choosing $y, x, -2z$ as multipliers, each fraction equals

$$\frac{ydx + xdy - 2zdz}{xy + 2yz + 4x^2z - xy - 4x^2z - 2yz}$$

$$\therefore ydx + xdy - 2zdz = 0$$

$$\therefore d(xy) - 2zdz = 0$$

Integrating we get, $xy + \frac{-2z^2}{2} = c_2$

$$xy - z^2 = c_2 \quad (15)$$

From equations (14) and (15), the required general solution is

$$\Phi(-x^2 + y + z, xy - z^2)$$

where ϕ is arbitrary function.

General Solution of Lagrange's Equation

Problem 6:

Solve the PDE of $(y - zx) p + (x + yz) q = x^2 + y^2$

Solution

It is of the form

$$P p + Q q = R$$

Lagrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{(y - zx)} = \frac{dy}{(x + yz)} = \frac{dz}{(x^2 + y^2)} \quad (16)$$

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Solution of Problem 6 Continue...

Choosing $y, x, -1$ as multipliers, each fraction equals

$$\begin{aligned} & \frac{ydx + xdy - dz}{y^2 - xyz + x^2 + xyz - x^2 - y^2} \\ \therefore ydx + xdy - dz = 0 & \therefore d(xy) - dz = 0 \text{ Integrating we get,} \\ \therefore xy - z &= c_1 \end{aligned}$$

$$xy - z = c_1 \quad (17)$$

General Solution of Lagrange's Equation

Solution of Problem 6 Continue...

Choosing $x, -y, z$ as multipliers, each fraction equals

$$= \frac{xdx - ydy + zdz}{xy - x^2z - xy - y^2z + x^2z + y^2z}$$

$$\therefore xdx - ydy + zdz = 0$$

Integrating we get, $\frac{x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2} = c_2$

$$x^2 - y^2 + z^2 = c_2 \quad (18)$$

From equations (17) and (18), the required general solution is

$$\Phi(xy - z, x^2 - y^2 + z^2)$$

where ϕ is arbitrary function.

General Solution of Langrange's Equation

Problem 7:

Solve the PDE of $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x + y)z$

Solution

Let $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$

It is of the form

$$P p + Q q = R$$

Langrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x + y)z} \quad (19)$$

General Solution of Langrange's Equation

Solution of Problem 7 Continue...

Consider first two ratio of equation (19)

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\therefore x^{-2}dx = y^{-2}dy$$

Integrating we get,

$$\therefore \frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + c_1$$

$$\therefore \frac{1}{y} - \frac{1}{x} = c_1$$

$$\frac{x - y}{xy} = c_1 \quad (20)$$

General Solution of Lagrange's Equation

Solution of Problem 7 Continue...

Choosing $\frac{1}{x}, \frac{1}{y}, -\frac{1}{z}$ as multipliers, each fraction equals

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{x + y - x - y} \implies \frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz = 0$$

Integrating we get,

$$\therefore \ln x + \ln y - \ln z = \ln c_2 \implies \therefore \ln \frac{xy}{z} = \ln c_2$$

$$\frac{xy}{z} = c_2 \quad (21)$$

From equations (20) and (21), the required general solution is

$$\Phi\left(\frac{x-y}{xy}, \frac{xy}{z}\right)$$

General Solution of Lagrange's Equation

Problem 8:

Find the general solution / complete integral of PDE
 $z(x p - y q) = y^2 - x^2$

Solution

Given $xz p - yz q = y^2 - x^2$
 It is of the form $P p + Q q = R$
 Lagrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{xz} = \frac{dy}{-yz} = \frac{dz}{y^2 - x^2} \quad (22)$$

General Solution of Langrange's Equation

Solution of Problem 8 Continue...

Consider first two ratio of equation (22)

$$\frac{dx}{xz} = \frac{dy}{-yz}$$

$$\therefore \frac{dx}{x} = \frac{dy}{-y}$$

Integrating we get,

$$\therefore \ln x = -\ln y + \ln c_1$$

$$\therefore \ln x + \ln y = \ln c_1$$

$$\therefore \ln xy = \ln c_1$$

Taking antilog, we get

$$xy = c_1$$

(23)

General Solution of Lagrange's Equation

Solution of Problem 8 Continue...

Choosing $(x + y)$, $(x + y)$, z as multipliers, fraction equals

$$= \frac{(x + y)dx + (x + y)dy + zdz}{x^2z + xyz - xyz - y^2z + y^2z - x^2z}$$

$$xdx + ydy + (ydy + xdy) + zdz = 0$$

Integrating we get,

$$\frac{x^2}{2} + \frac{y^2}{2} + xy + \frac{z^2}{2} = c_2$$

$$x^2 + y^2 + 2xy + z^2 = c_2 \quad (24)$$

From equations (23) and (24), the required general solution is

$$\Phi(xy, x^2 + y^2 + 2xy + z^2)$$

where ϕ is arbitrary function.

General Solution of Lagrange's Equation

Problem 9:

Find the general solution / complete integral of PDE
 $y^2 p - xy q = x(z - 2y)$

Solution

Given $y^2 p - xy q = x(z - 2y)$

It is of the form $P p + Q q = R$

Lagrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)} \quad (25)$$

General Solution of Langrange's Equation

Solution of Problem 9 Continue...

Consider first two ratio of equation (25)

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\therefore \frac{dx}{y} = \frac{dy}{-x}$$

$\therefore xdx = -ydy$ Integrating we get,

$$\therefore \frac{x^2}{2} = -\frac{y^2}{2} + c_1$$

$$\therefore \frac{x^2}{2} + \frac{y^2}{2} = c_1$$

Taking antilog, we get

$$x^2 + y^2 = c_1 \quad (26)$$

General Solution of Lagrange's Equation

Solution of Problem 9 Continue...

Choosing $2x, z, y$ as multipliers, fraction equals

$$\frac{2xdx + zdy + ydz}{2xy^2 - xyz + xyz - 2xy^2}$$

$$\therefore 2xdx + zdy + ydz = 0 \implies 2xdx + d(yz) = 0$$

Integrating we get,

$$2\frac{x^2}{2} + yz = c_2$$

$$x^2 + 2yz = c_2 \quad (27)$$

From equations (26) and (27), the required general solution is

$$\Phi(x^2 + y^2, x^2 + 2yz)$$

where ϕ is arbitrary function.

General Solution of Lagrange's Equation

Problem 10:

Solve the PDE of $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x + y + z$

Solution

Given $p + q = x + y + z$

It is of the form $P p + Q q = R$

Lagrange's Auxiliary Equation's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x + y + z} \quad (28)$$

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Solution of Problem 10 Continue...

Consider first two ratio of equation (28)

$$\frac{dx}{1} = \frac{dy}{1}$$

$$\therefore dx = dy$$

Integrating we get,

$$\therefore x = y + c_1$$

$$x - y = c_1 \quad (29)$$

General Solution of Lagrange's Equation

Solution of Problem 10 Continue...

Choosing 1, 1, 1 as multipliers, fraction equals

$$\frac{dx}{1} = \frac{dx + dy + dz}{2 + x + y + z}$$

Integrating we get, $x = \ln(2 + x + y + z) + \ln c_2$

$$x = \ln [c_2(2 + x + y + z)] \implies e^x = c_2(2 + x + y + z)$$

$$\frac{e^x}{(2 + x + y + z)} = c_2 \quad (30)$$

From equations (28) and (29), the required general solution is

$$\Phi\left(x - y, \frac{e^x}{(2 + x + y + z)}\right)$$

where ϕ is arbitrary function.