

# Learn Mathematics Free With Us

WELCOME TO

An Online Tutoring Programme for B.E. / B.Sc. students to learn Mathematics

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

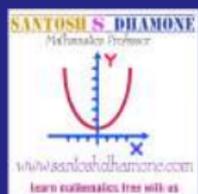
## Unit I : First Order Partial Differential Equation

Subject Teacher  
Santosh Dhamone

Assistant Professor in Mathematics  
Art's Commerce and Science College, Onda  
Tal:- Vikramgad, Dist:- Palghar

*ssdhamone@acscollegeonde.ac.in*

January 30, 2024

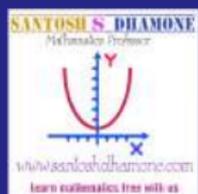


# Contents

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

Curves and Surfaces  
Genesis of first order PDE  
Classification of first order PDE  
Classification of integrals  
The Cauchy problem  
Linear Equation of first order  
Lagrange's equation  
Pfaffian differential equations

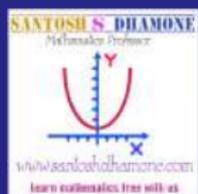


# PDE: Unit I : Partial Differential Equations

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

- **Parametrically Defined Curve in  $\mathbb{R}^2$ :**  
Parametrically Defined Curve in  $\mathbb{R}^2$  is the plane curve  $C$  given by a function  $f : D \rightarrow \mathbb{R}^2, f(t) = (x(t), y(t))$  where  $D \subseteq \mathbb{R}$ .  
Usually, we express this by simply saying that  $C$  is the (parametrically defined) curve given by  $(x(t), y(t)), t \in D$ . For example, the rectangular hyperbola is the curve  $(t, \frac{1}{t}), t \in \mathbb{R} \setminus \{0\}$ .
- **Parametrically Defined Curve in  $\mathbb{R}^3$ :**  
A parametrically defined curve  $C$  in  $\mathbb{R}^3$  is given by  $(x(t), y(t), z(t)), t \in D$  where  $D \subseteq \mathbb{R}$ .
- **Parametrically Defined Surface  $S$  in  $\mathbb{R}^3$**  is given by a function  $f : D \rightarrow \mathbb{R}^3, f(u, v) = (x(u, v), y(u, v), z(u, v))$  where  $x, y, z$  are real valued functions on  $D$ , that is,  $x, y, z : D \rightarrow \mathbb{R}$ .



# PDE: Unit I : Chain Rules

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

We will need the following Chain Rules.

- If  $z = g(v)$  and  $v = f(x, y)$  and then  $z$  is a function of  $(x, y)$ , and

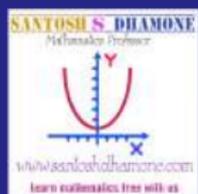
$$\frac{\partial z}{\partial x} = \frac{dz}{dv} \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{dz}{dv} \frac{\partial v}{\partial y}$$

- If  $z = f(x, y)$  and  $x = x(t), y = y(t)$ , then  $z$  is a function of  $t$ , and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

- If  $z = f(u, v)$  and if  $u = u(x, y), v = v(x, y)$ , then  $z$  is a function of  $(x, y)$ , and

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$



# PDE: Unit I : Useful Notations

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

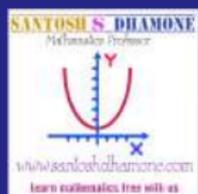
- If  $z$  is a function of  $(x, y)$  then  $\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q.$

- If  $u = u(x, y)$  and  $v = v(x, y)$  then

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \text{Jacobian matrix of } u \text{ and } v \text{ w.r.t. } x \text{ and } y.$$

- $\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \text{Jacobian of } u \text{ and } v \text{ w.r.t. } x \text{ and } y.$

- Notation  $\frac{\partial(u, v)}{\partial(x, y)} = \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$



# PDE: Unit I : Definition of PDE, Order and Degree

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

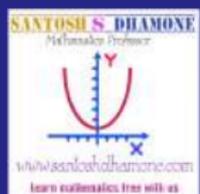
Subject Teacher  
Santosh Dhamone

- An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a **partial differential equation**.
- General format  $f(x, y, z, p, q) = 0$
- For example  $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$

That is,

$$p + q = z + xy$$

- **Order of a partial differential equation** is defined as the order of the highest order partial derivative occurring in the partial differential equation.
- **Degree of a partial differential equation** is defined as the power of the highest order partial derivative occurring in the partial differential equation after the equation has been made free from radicals and fractions so far as derivatives are concerned.



# Unit I : Classification of First Order PDE

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

A first order partial differential equation  $f(x, y, z, p, q) = 0$  is known as

Linear eqn	Semi-linear	Quasi-linear
linear in $p, q$ and $z$	linear in $p, q$	linear in $p, q$
$P(x, y)p + Q(x, y)q$ $= R(x, y)z + S(x, y)$	$P(x, y)p + Q(x, y)q$ $= R(x, y, z)$	$P(x, y, z)p + Q(x, y, z)q$ $= R(x, y, z)$
For example $yx^2 p + xy^2 q$ $= x + y + x^2 y^2 z$	For example $yx^2 p + xy^2 q$ $= x + y + x^2 y^2 z^2$	For example $yx^2 z p + xy^2 z q$ $= x + y$

# Unit I : Obtaining a PDE

My Inspiration  
 Late. Shival  
 Dhamone  
 and  
 Shri. V. G. Patil  
 Saheb

Subject Teacher  
 Santosh Dhamone

## Theorem 0.1

**TYPE 1** The elimination of arbitrary function  $\phi$  from the equation  $\phi(u, v) = 0$ , where  $u$  and  $v$  are functions of  $x, y$  and  $z$  ( $z$  is assumed to be a function of  $x$  and  $y$ ), gives the partial differential equation

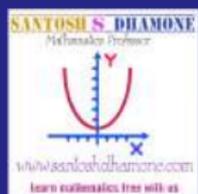
$$\frac{\partial(u, v)}{\partial(y, z)} * p + \frac{\partial(u, v)}{\partial(z, x)} * q = \frac{\partial(u, v)}{\partial(x, y)}.$$

Obtain a pde by eliminating the arbitrary function  $\phi$  from  $\phi(x + y + z, x^2 + y^2 - z^2) = 0$

Let  $u(x, y, z) = x + y + z, v(x, y, z) = x^2 + y^2 - z^2$ .

$\frac{\partial(u, v)}{\partial(y, z)}$	$\frac{\partial(u, v)}{\partial(z, x)}$	$\frac{\partial(u, v)}{\partial(x, y)}$
$\det \begin{pmatrix} 1 & 1 \\ 2y & -2z \end{pmatrix}$	$\det \begin{pmatrix} 1 & 1 \\ -2z & 2x \end{pmatrix}$	$\det \begin{pmatrix} 1 & 1 \\ 2x & 2y \end{pmatrix}$
$-2(y + z)$	$2(x + z)$	$2(y - x)$

The required pde is  $(y + z) p - (x + z) q = x - y$



# Unit I : Obtaining a PDE

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Theorem 0.2

**TYPE 2:** Let  $v = v(x, y)$  and  $z = f(v)$  then  $\frac{\partial(z, v)}{\partial(x, y)} = 0$  is a first order pde for  $z$ . That is, the pde is  $\left| \begin{array}{c} p \\ v_x \end{array} \quad \begin{array}{c} q \\ v_y \end{array} \right| = 0$

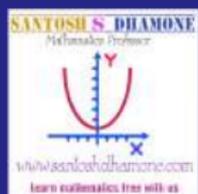
**Example:** Eliminate arbitrary function  $f$  from  $z = f(x^2 - y^2)$  and obtain the corresponding pde.

Let  $v = x^2 - y^2$ . So  $v_x = 2x, v_y = -2y$ .

The partial differential is given by  $\frac{\partial(z, v)}{\partial(x, y)} = 0$

That is,  $v_y p - v_x q = 0$

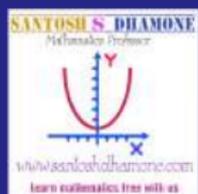
Substituting we get,  $(-2y) p - (2x) q = 0$ . The p.d.e. is  
 $y p + x q = 0$ .



# Unit I : Obtaining a PDE

Using theorem 0.2 we can prove following results.

Equation	pde
$z = f(x^2 - y^2)$	$y p + x q = 0$
$z = f(x^2 + y^2)$	$y p - x q = 0$
$lx + my + nz = f(x^2 + y^2 + z^2)$	$\frac{l + np}{m + nq} = \frac{x + zp}{y + zq}$
$z = e^{ax+by} f(ax - by)$	$b p + a q = 2ab z$
$z = x^n f\left(\frac{y}{x}\right)$	$x p + y q = nz$



# Unit I : Obtaining a PDE

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

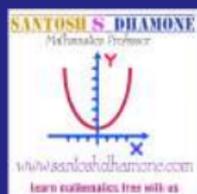
**TYPE 3:** Obtain a pde by the elimination of arbitrary constants:

Consider the equation

$$F(x, y, z, a, b) = 0$$

where  $a$  and  $b$  denote arbitrary constats.

- **Case 1:** No. of arbitrary constants  $<$  No. of independent variables.  
For example:  $z = ax + y$ . So,  $p = a, q = 1$ .  
Substituting  $a = p$  in the equation, we get one pde  $z = xp + y$ .  
Note that  $q = 1$  is also a pde of  $z = ax + y$ .
- **Case 2:** No. of arbitrary constants  $=$  No. of independent variables.  
then the elimination gives rise to a unique partial differential equation of order one.
  - For example,  $az + b = a^2x + y$ .  
 $\implies ap = a^2$  and  $aq = 1 \implies a^2 p * q = a^2$
  - The pde is  $pq = 1$ . Note: pde is not a linear diff. eqn.
- **Case 3:** No. of arbitrary constants  $>$  No. of independent variables.  
For example  $z = ax + by + cxy$ . The elimination of arbitrary constants leads to a pde of order usually greater than one.



# Unit I : Integral Surfaces

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

- Consider a first order partial differential equation in two unknowns  $x$  and  $y$  given be

$$f(x, y, z, p, q) = 0 \quad (1)$$

The solution  $z = F(x, y; a, b)$  of (1) represents a surface in  $(x, y, z)$  space.

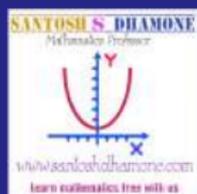
This surface is called an **integral surface** of the partial differential equation (1).

- A two parameter family of solutions  $z = F(x, y; a, b)$  of the equation

$$f(x, y, z, p, q) = 0.$$

is called a **complete integral** of the equation  $f(x, y, z, p, q) = 0$  if the rank of the matrix

$$M = \begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix} \text{ is two.}$$



# Unit I : Example on Complete Integral

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Example 0.3

Consider  $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$ .

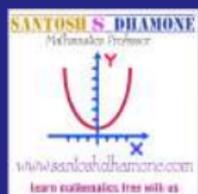
Show that the two parameter family of  $z = F(x, y; a, b)$  given by  $z = ax + by + a^2 + b^2$  is a complete integral.

Solution:

$$z = ax + by + a^2 + b^2 \implies p = z_x = a, q = z_y = b.$$

$$\begin{aligned} \text{L.H.S} &= z - px - qy - p^2 - q^2 \\ &= ax + by + a^2 + b^2 - ax - by - a^2 - b^2 \\ &= 0 \end{aligned}$$

Hence  $z = ax + by + a^2 + b^2$  is a solution of  $z - px - qy - p^2 - q^2 = 0$ .



# Unit I : Example on Complete Integral

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

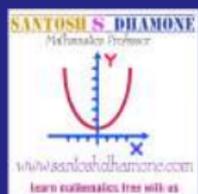
To show that  $z = ax + by + a^2 + b^2$  is a complete integral of  $z - px - qy - p^2 - q^2 = 0$ .

$$F(x, y; a, b) = ax + by + a^2 + b^2$$

$$\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix} = \begin{pmatrix} x + 2a & 1 & 0 \\ y + 2b & 0 & 1 \end{pmatrix}$$

Rank of the above matrix is 2.

Hence  $z = ax + by + a^2 + b^2$  is a complete integral of  $z - px - qy - p^2 - q^2 = 0$ .



# Unit I : Envelope of one parameter family of surfaces

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

- Let  $S_a$  be a family of one parameter surfaces  $z = F(x, y; a)$  where  $a$  is the parameter. Consider the following system of equations.

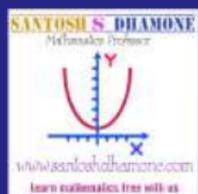
$$\begin{aligned}z &= F(x, y; a), \\ 0 &= F_a(x, y; a).\end{aligned}$$

The **envelope**  $E$  of  $S_a$ , if exists, is defined as the set of all  $(x, y, z) \in \mathbb{R}^3$  satisfying the above system of equations for some value of the parameter  $a$ .

- For a fixed value of  $a$ , these two equations determine a curve  $C_a$ . The envelope  $E$  of the family of surfaces  $S_a$  is the union of all these curves  $C_a$ .
- The envelope  $E$  of the family of surfaces  $S_a$ , is obtained by eliminating  $a$  between

$$z = F(x, y; a), \tag{2}$$

$$0 = F_a(x, y; a). \tag{3}$$



# Unit I : Classification of Integral Surfaces

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

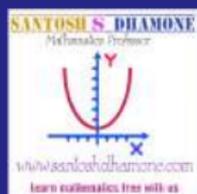
## Lemma 0.4

Consider the partial differential equation  $f(x, y, z, p, q) = 0$

Let  $S_a$  be a one parameter family of solutions  $z = F(x, y; a)$  where  $a$  is the parameter of (\*).

Then the envelope of this family, if it exists, is also a solution of

$$f(x, y, z, p, q) = 0.$$



# Unit I : Envelope of one parameter family of surfaces

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

- Let  $S_{a,b}$  be a family of surfaces of two parameters  $a$  and  $b$  given by  $z = F(x, y; a, b)$

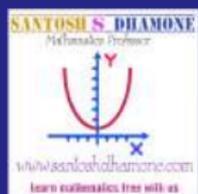
Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be any function.

Let  $S_{a,\phi}$  be the one-parameter family of surfaces given by  $z = F(x, y, a, \phi(a))$ .

Consider the following system of equations.

$$\begin{aligned}z &= F(x, y; a, \phi(a)), \\ 0 &= F_a + F_b \phi'(a).\end{aligned}$$

The envelope of  $S_{a,\phi}$ , if exists, is defined as the set of all  $(x, y, z) \in \mathbb{R}^3$  satisfying the above system of equations for some value of the parameter  $a$ .



# Unit I : General Integral Solution

My Inspiration  
Late. Shivlal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

- Let  $S_{a,b}$  be a two parameter family  $z = F(x, y, a, b)$  of complete solutions of  $f(x, y, p, q) = 0$  where  $a, b$  are the parameters. Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be any function.

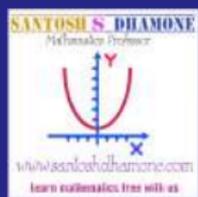
Let  $S_{a,\phi}$  be a family of the surfaces  $z = F(x, y; a, \phi(a))$ .

Then the envelope of  $S_{a,\phi}$  is also a solution of  $f(x, y, p, q) = 0$ .

This solution is called a **General integral** of  $f(x, y, z, p, q) = 0$ .

- When a particular function  $\phi$  is used, we obtain a **particular integral** of the partial differential equation.

Different choices of  $\phi$  may give different particular solutions of the partial differential equation.



# Unit I : Example on Particular Integral

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Example 0.5

Consider  $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$ .

Given that  $z = F(x, y; a, b) = ax + by + a^2 + b^2$  is a complete integral.

If  $b = \sqrt{1 - a^2}$  then find the particular integral.

The envelope  $E$  of the family  $z = F(x, y, a, \phi(a))$  is obtained by eliminating  $a$  between  $z = F(x, y, a, \phi(a))$  and

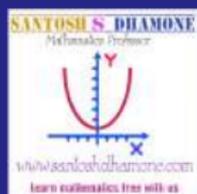
$$F_a(x, y, a, b) + F_b(x, y, a, b)\phi'(a) = 0$$

$$b = \sqrt{1 - a^2} \implies \phi(a) = \sqrt{1 - a^2}. \text{ So } \phi'(a) = \frac{-a}{\sqrt{1 - a^2}}$$

$$z = F(x, y; a, b) = ax + by + a^2 + b^2$$

$$F_a + F_b * \phi'(a) = 0 \implies (x + 2a) + (y + 2b) * \frac{-a}{\sqrt{1 - a^2}} = 0$$

Hence  $a = \frac{x}{\sqrt{x^2 + y^2}}$ . Putting this in  $z = ax + by + a^2 + b^2$ , we get,



# Unit I : Example on Particular Integral

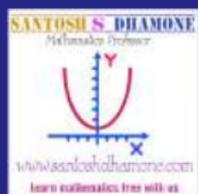
My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

$$\begin{aligned}z &= \frac{x^2}{\sqrt{x^2 + y^2}} + \sqrt{\frac{y^2}{x^2 + y^2}} * y + 1 \\z &= \frac{x^2}{\sqrt{x^2 + y^2}} + \frac{y^2}{\sqrt{x^2 + y^2}} + 1 \\&= \sqrt{x^2 + y^2} + 1\end{aligned}$$

Hence the envelope is given by  $z = \sqrt{x^2 + y^2} + 1$ .

The particular integral is  $z = \sqrt{x^2 + y^2} + 1$ .



# Unit I : Envelope of two parameters family of surfaces and Singular Integral

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

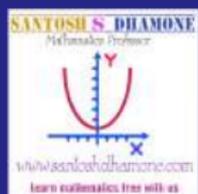
- Let  $S_{a,b}$  be a family of two parameter surfaces  $z = f(x, y; a, b)$  where  $a, b$  are the parameters. Consider the following system of equations.

$$z = F(x, y; a, b),$$

$$0 = F_a(x, y; a, b),$$

$$0 = F_b(x, y; a, b).$$

The envelope  $E$  of  $S_{a,b}$ , if exists, is defined as the set of all  $(x, y, z) \in \mathbb{R}^3$  satisfying the above system of equations for some values of the parameters  $a$  and  $b$ .



# Unit I : Singular Integral

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Lemma 0.6

Let  $S_{a,b}$  be a two parameter family of complete integrals  $z = F(x, y; a, b)$  of  $f(x, y, z, p, q) = 0$  where  $a, b$  are the parameters. Then the envelope of  $S_{a,b}$  is also a solution of  $f(x, y, p, q) = 0$ .

(This is Lemma no. 1.3.2 in our syllabus)

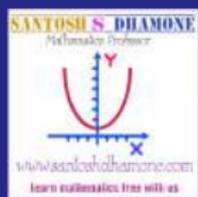
- This solution is called a **singular** integral of  $f(x, y, z, p, q) = 0$ .

## Example 0.7

Consider  $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$ .

Given that  $z = F(x, y; a, b) = ax + by + a^2 + b^2$  is a complete integral.

Find the singular integral.



# Unit I : Example: Finding Singular Integral

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

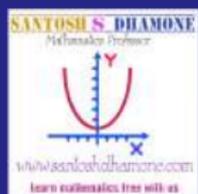
- For singular integral, we take  $z = F(x, y; a, b)$  and then we eliminate  $a$  and  $b$  using the equations  $F_a(x, y; a, b) = 0$  and  $F_b(x, y; a, b) = 0$ . Here,

$$\begin{aligned}z = F(x, y; a, b) &= ax + by + a^2 + b^2 \\F_a &= x + 2a \\F_b &= y + 2b \\F_a = 0, F_b = 0 &\implies a = -\frac{x}{2}, \quad b = -\frac{y}{2}\end{aligned}$$

Substituting these values in  $z = F(x, y; a, b) = ax + by + a^2 + b^2$ , we get

$$z = -\frac{x^2 + y^2}{4}$$

That is,  $4z = -(x^2 + y^2)$



# Unit I : Another method to Find Singular Integral

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Lemma 0.8

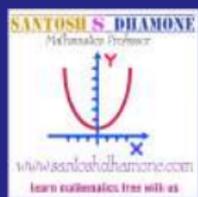
Let  $z = F(x, y; a, b)$  be a complete integral of  $f(x, y, z, p, q) = 0$  and  $z = F(x, y, a(x, y), b(x, y))$  be the singular integral of  $f(x, y, z, p, q) = 0$ .

Then the singular integral satisfies the equations

$$f(x, y, z, p, q) = 0,$$

$$f_p(x, y, z, p, q) = 0,$$

$$f_q(x, y, z, p, q) = 0.$$



# Unit I : Another method to Find Singular Integral

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Example 0.9

Consider  $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$ .

Given that  $z = F(x, y; a, b) = ax + by + a^2 + b^2$  is a complete integral.

Find the singular integral using the above lemma.

We know that singular integral satisfies:

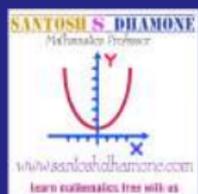
$$f(x, y, z, p, q) = 0 \implies z - px - qy - p^2 - q^2 = 0,$$

$$f_p(x, y, z, p, q) = 0 \implies x - 2p = 0,$$

$$f_q(x, y, z, p, q) = 0 \implies -y - 2q = 0.$$

This implies  $p = -\frac{x}{2}, q = -\frac{y}{2}$ .

Hence the singular solution is  $z = -\frac{x^2 + y^2}{4}$



# Unit I : Cauchy Problem

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

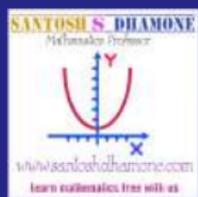
## The Cauchy Problem

- Given a first order partial differential equation and a curve in space, the Cauchy problem is to find an integral surface (i.e. a solution) of the given partial differential equation which contains the given curve. In other words, given a partial differential equation (not necessarily non-linear)

$$f(x, y, z, p, q) = 0$$

and a curve  $x = x(s), y = y(s), z = z(s), s \in [a, b]$ ,

the Cauchy problem is to find a solution  $z = z(x, y)$  of the pde such that  $z(s) = z(x(s), y(s))$  for all  $s \in [a, b]$ .  
(Note: We will be studying this in unit II in detail.)



# Unit I : General Solutions of Quasi linear equations or Lagrange's equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Theorem 0.10

The general solution of the Lagrange equation

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z), \quad (*)$$

where  $P, Q$  and  $R$  are continuously differentiable functions on the domain  $D \subseteq \mathbb{R}^3$  is  $\phi(u, v) = 0$  where  $\phi$  is an arbitrary function and

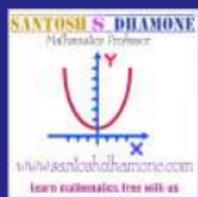
$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2$$

are two independent solutions of  $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ .

(Lagrange's auxiliary equations of  $(*)$ )

The general solution (or integral) of (1) is written in one of the following three equivalent forms:

$$\phi(u, v) = 0, \quad u = G(v) \quad \text{or} \quad v = H(u)$$



# Unit I : General Solution of Lagrange's Equation (more no. of ind. variables)

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Theorem 0.11

A general solution of the quasi-linear partial differential equation

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \cdots + P_n \frac{\partial z}{\partial x_n} = R.$$

where  $P_1, P_2, \dots, P_n, R$  are continuously differentiable functions of  $x_1, x_2, \dots, x_n$  and  $z$ , not simultaneously zero,

is the relation  $\phi(u_1, u_2, \dots, u_n) = 0$  where  $\phi$  is an arbitrary differentiable function and  $u_1(x_1, x_2, \dots, x_n, z) = c_1, u_2(x_1, x_2, \dots, x_n, z) = c_2, \dots, u_n(x_1, x_2, \dots, x_n, z) = c_n$  are **independent** solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R}.$$



# Unit I :Type 1: Solving Lagrange's Equation

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

Solve the pde of  $\frac{y^2 z}{x} p + xz q = y^2$

Lagrange's auxiliary eqns are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \quad (1).$$

Taking the first two fractions, from (1)

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz}$$

$$x^2 dx = y^2 dy$$

$$x^3 = y^3 + c_1 \quad (2)$$

Taking the first and the last from (1).

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dz}{y^2}$$

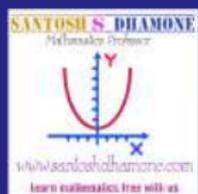
$$x dx = z dz$$

$$x^2 = z^2 + c_2 \quad (3).$$

From (2) and (3), the required general solution is

$$\phi(x^3 - y^3, x^2 - z^2) = 0.$$

Another form of the general integral is  $G(x^3 - y^3) = x^2 - z^2$ .



# Unit I : Type 2: Solving Lagrange's Equation

My Inspiration  
Late. Shivlal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

Solve the pde  $p + 3q = 5z + \tan(y - 3x)$

Lagrange's auxiliary eqns are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{\tan(y - 3x)} \quad (1).$$

Taking the first two fractions, from (1)

$$\frac{dx}{1} = \frac{dy}{3}$$

$$y - 3x = c_1 \quad (2).$$

$c_1$  denotes a constant.

Taking the first and the last from (1).

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$$

$$x = \frac{1}{5} \ln(5z + \tan c_1) = c_2$$

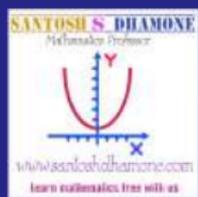
$$5x - \ln(5z + \tan c_1) = c_2 \quad (3).$$

$(c_2$  denotes a constant)

From (2) and (3), the required general solution is

$$\phi(y - 3x, 5x - \ln(5z + \tan c_1)) = 0$$

where  $\phi$  is an arbitrary function.



# Unit I : Type 3: Solving Lagrange's Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb  
  
Subject Teacher  
Santosh Dhamone

Let  $P_1, Q_1$  and  $R_1$  be functions of  $x, y$  and  $z$ .

Then each fraction in Lagrange's auxiliary eqns

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \text{ is equal to}$$

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \quad (*)$$

If  $P_1 P + Q_1 Q + R_1 R = 0$ , then the numerator of (\*) is also 0. This gives

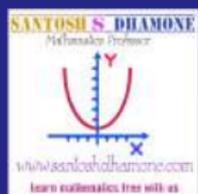
$$P_1 dx + Q_1 dy + R_1 dz = 0 \quad (**)$$

This can be integrated to get  $u_1(x, y, z) = c_1$ . This method may be repeated to get another integral  $u_2(x, y, z) = c_2$ .  $P_1, Q_1, R_1$  are called multipliers.

Solve the pde  $yz \frac{b-c}{a} p + zx \frac{c-a}{b} q = \frac{a-b}{c} xy$ .

Lagrange's auxiliary eqns are

$$\frac{a dx}{yz(b-c)} = \frac{b dy}{zx(c-a)} = \frac{c dz}{xy(a-b)} \quad \dots (1).$$



# Unit I :Type 3: Solving Lagarange's Equation

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

Choosing  $x, y, z$  as multipliers,  
each fraction equals

$$= \frac{ax \, dx + by \, dy + cz \, dz}{0}$$

$$\Rightarrow ax \, dx + by \, dy + cz \, dz = 0.$$

Integrating

$$a \frac{x^2}{2} + b \frac{y^2}{2} + c \frac{z^2}{2} = c_1$$

$$\Rightarrow ax^2 + by^2 + cz^2 = c_1 \quad \dots (2)$$

( $c_1$  being arbitrary constant).

Now, choosing  $ax, by$  and  $cz$  as  
multipliers for eqn (1), we get

$$= \frac{a^2x \, dx + b^2y \, dy + c^2z \, dz}{xyz(a(b-c) + b(c-a) + c(a-b))}$$

$$= \frac{a^2x \, dx + b^2y \, dy + c^2z \, dz}{0}$$

$$\Rightarrow a^2x \, dx + b^2y \, dy + c^2z \, dz = 0$$

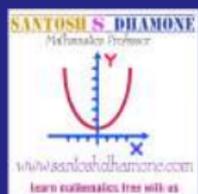
Integrating,

$$a^2x^2 + b^2y^2 + c^2z^2 = c_2 \quad \dots (3)$$

From (2) and (3), the required general  
solution is

$$\phi(ax^2 + by^2 + cz^2, a^2x^2 + b^2y^2 + c^2z^2) = 0.$$

where  $\phi$  is an arbitrary function.



# Unit I :Type 4: Solving Lagarange's Equation

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

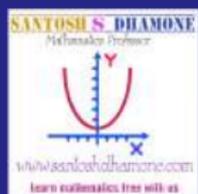
Let  $P_1, Q_1$  and  $R_1$  be functions of  $x, y$  and  $z$ .

Then all fractions in  $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$  are equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \quad (**).$$

Suppose the numerator is the exact differential of the denominator of (\*\*).

Then (\*\*) can be combined with a suitable fraction in (\*) to give an integral.



# Unit I :Type 4: Solving Lagarange's Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

$$(y + z) p + (z + x) q = x + y$$

Lagrange's auxiliary eqns are

$$\frac{dx}{(y + z)} = \frac{dy}{(z + x)} = \frac{dz}{x + y} \dots (1).$$

Choosing 1, -1, 0 as multipliers,  
each fraction equals

$$= \frac{dx - dy}{y - x} = -\frac{d(x - y)}{x - y} \dots (2).$$

Choosing 0, 1, -1 as multipliers,  
each fraction equals

$$= \frac{dy - dz}{z - y} = -\frac{d(y - z)}{y - z} \dots (3).$$

Choosing 1, 1, 1 as multipliers, each  
fraction equals

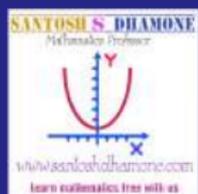
$$= \frac{dx + dy + dz}{y + z + z + x + x + y} = \frac{dx + dz}{2(x + y + z)} \dots (4).$$

From (2), (3) and (4), we have,

$$-\frac{d(x - y)}{x - y} = -\frac{d(y - z)}{y - z} = \frac{dx + dy + dz}{2(x + y + z)} \dots (5).$$

Taking the first two fractions of (5)

$$\frac{d(x - y)}{x - y} = \frac{d(y - z)}{y - z}$$



# Unit I :Type 4: Solving Lagarange's Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

$$\text{Integrating, } \implies \frac{x - y}{y - z} = c_1 \quad \dots (6).$$

Taking the first and third fractions of (5)

$$\frac{d(x - y)}{x - y} = \frac{dx + dy + dz}{2(x + y + z)}$$

$$\text{Integrating, } -2 \ln(x - y) = \ln(x + y + z) + \ln C_2$$

$$\ln(x + y + z) + 2 \ln(x - y) = - \ln C_2$$

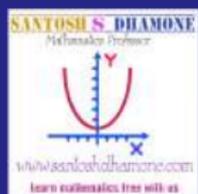
$$\ln(x + y + z)(x - y)^2 = \ln c_2$$

$$(x + y + z)(x - y)^2 = c_2 \quad (7).$$

From (6) and (7), the required general solution is

$$\phi \left( \frac{x - y}{y - z}, (x + y + z)(x - y)^2 \right) = 0.$$

where  $\phi$  is an arbitrary function.



# Unit I : Pfaffian Differential Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

- By a **Pfaffian differential equation**, we mean a differential equation of the form

$$F_1(x_1, x_2, \dots, x_n)dx_1 + \dots + F_n(x_1, x_2, \dots, x_n)dx_n = 0 \quad (*)$$

where  $F_i$ 's,  $1 \leq i \leq n$  are continuous functions.

The expression on the LHS is called a **Pfaffian differential form**.

- A Pfaffian differential form

$$F_1(x_1, x_2, \dots, x_n)dx_1 + \dots + F_n(x_1, x_2, \dots, x_n)dx_n$$

is said to be **exact** if we can find a **continuously differentiable function**  $u(x_1, x_2, \dots, x_n)$  such that

$$du = F_1(x_1, x_2, \dots, x_n)dx_1 + \dots + F_n(x_1, x_2, \dots, x_n)dx_n.$$



# Unit I : Pfaffian Differential Equation

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

- A Pfaffian differential equation

$$F_1(x_1, x_2, \dots, x_n)dx_1 + \dots + F_n(x_1, x_2, \dots, x_n)dx_n = 0 \quad (**)$$

is said to be **exact** if the Pfaffian differential form on the LHS of the equation is exact.

- That is, the Pfaffian differential equation (\*\*) is said to be exact if we can find a **continuously differentiable function**  $u(x_1, x_2, \dots, x_n)$  such that

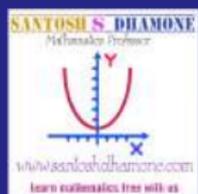
$$du = F_1(x_1, x_2, \dots, x_n)dx_1 + \dots + F_n(x_1, x_2, \dots, x_n)dx_n.$$

- The function  $u(x_1, x_2, \dots, x_n) = c$ , is called the **integral** of the corresponding **Pfaffian differential equation**.
- The Pfaffian differential equation (\*\*) is said to be **integrable** if there exists a **non-zero differentiable function**  $\mu(x_1, x_2, \dots, x_n)$  such that the Pfaffian differential form

$$\mu (F_1(x_1, x_2, \dots, x_n)dx_1 + \dots + F_n(x_1, x_2, \dots, x_n)dx_n)$$

is exact.

- The function  $\mu(x_1, x_2, \dots, x_n)$  is called an **integrating factor** of (\*\*).



# Unit I : Pfaffian Differential Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Theorem 0.12

There always exists an **integrating factor** for a Pfaffian differential equation in two variables ( $P(x, y) dx + Q(x, y) dy = 0$ ).

## Lemma 0.13

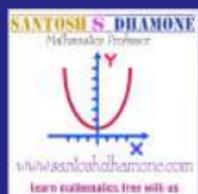
Let  $u(x, y) = c_1$  and  $v(x, y) = c_2$  be two functions of  $x$  and  $y$  such that

$$\frac{\partial v}{\partial y} \neq 0.$$

If, further

$$\frac{\partial(u, v)}{\partial(x, y)} = 0,$$

then there exists a relation  $F(u, v) = 0$   
between  $u$  and  $v$  **not involving  $x$  and  $y$  explicitly.**



# Unit I : Pfaffian Differential Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

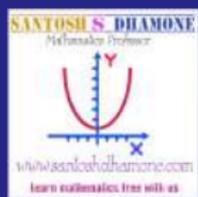
Subject Teacher  
Santosh Dhamone

Recall: If  $\bar{X} = (P, Q, R)$  then the curl of  $\bar{X}$  is defined by

$$\text{curl } \bar{X} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$$

- The definition of curl can be difficult to remember. To help with remembering, we use the following determinant formula.

$$\text{curl } \bar{X} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$



# Unit I : Pfaffian Differential Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Lemma 0.14

If  $\bar{X} = (P(x, y, z), Q(x, y, z), R(x, y, z))$  and  $\mu$  is an arbitrary nonzero differentiable function of  $x, y$  and  $z$  then

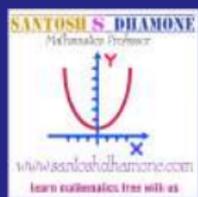
$$\bar{X} \cdot \text{curl } \bar{X} = 0 \quad \text{if and only if} \quad \mu \bar{X} \cdot \text{curl } (\mu \bar{X}) = 0.$$

## Theorem 0.15

**A necessary and sufficient condition** that the Pfaffian differential equation

$$\bar{X} \cdot \overline{dr} = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0 \quad (4)$$

to be integrable is that  $\bar{X} \cdot \text{curl } \bar{X} = 0$



# Unit I : Condition for Pfaffian Differential Equation to be exact

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

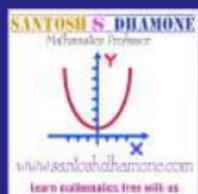
Subject Teacher  
Santosh Dhamone

## Remark 0.16

Necessary and sufficient condition for the Pfaffian differential equation  $\overline{X} \cdot \overline{dr} = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$  to be **exact** is

$$\text{curl } \overline{X} = \overline{0}$$

That is,  $R_y - Q_z = 0, P_z - R_x, Q_x - P_y = 0$



# Unit I : Example 1 of Pfaffian differential Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Example 0.17

Show that the following Pfaffian differential equation is exact and find its integral.  $y dx + x dy + 2z dz = 0$ .

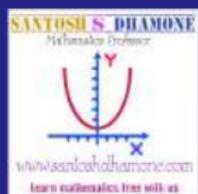
Here  $P = y, Q = x, R = 2z$ .

$$\text{curl } \bar{X} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 2z \end{pmatrix} = (0, 0, 0).$$

Clearly,  $y dx + x dy + 2z dz = d(xy + z^2)$

So,  $d(xy + z^2) = 0$ . This implies  $d(xy + z^2) = c$ .

Hence the integral is  $u(x, y, z) = xy + z^2 = c$ .



# Unit I : Example 2 of Pfaffian differential Equation

My Inspiration  
Late. Shivalal  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

## Example 0.18

Find the integral of  $yz \, dx + 2xz \, dy - 3xy \, dz = 0$

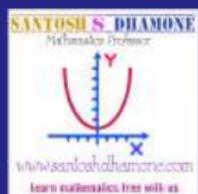
Here  $P = yz, Q = 2xz, R = -3xy$ .

$$\text{curl } \bar{X} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & -3xy \end{pmatrix} = \hat{i}(-3x-2x) - \hat{j}(-3y-y) + \hat{k}(2z-z).$$

$$\text{curl } \bar{X} = -5x \hat{i} + 4y \hat{j} + z \hat{k}$$

$$\bar{X} \cdot \text{curl } \bar{X} = -5xyz + 8xyz - 3xyz = 0$$

Hence given equation is integrable.



# Unit I : Example 2 of Pfaffian differential Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

Keep  $z$  as a constant and write the differential equation as follows:

$$yz \, dx + 2xz \, dy = 0$$

We find the solution of the above equation.

$$\frac{dx}{x} = -2 \frac{dy}{y}$$

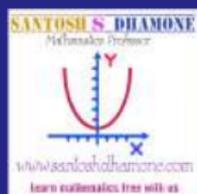
$x * y^2 = c_1$  where  $c_1$  is a constant and it may contain  $z$ .

So,  $U(x, y, z) = xy^2 = c_1$ .

**Now we find Integrating factor  $\mu$**

Consider equation  $\frac{\partial U}{\partial x} = \mu * P$       OR       $\frac{\partial U}{\partial y} = \mu * Q$

Here  $y^2 = \mu * yz \implies \mu = \frac{y}{z}$ .



# Unit I : Example 2 of Pfaffian differential Equation

My Inspiration  
Late. Shival  
Dhamone  
and  
Shri. V. G. Patil  
Saheb

Subject Teacher  
Santosh Dhamone

Now we find  $K = \left( \mu * R - \frac{\partial U}{\partial z} \right)$

$$K = \frac{y}{z} * (-3xy) - 0 = -\frac{3xy^2}{z} = -\frac{3U}{z}.$$

Substitute in the equation  $\frac{dU}{dz} + K = 0$

$$\text{This implies } \frac{dU}{dz} - \frac{3U}{z} = 0$$

We solve this equation.

Solution is  $U = cz^3$ .

This means  $xy^2 = cz^3$ .

Therefore the integral of the given Pfaffian equation is

$$u(x, y, z) = \frac{xy^2}{z^3} = c.$$