UNIT II : Compatible System of First Order Partial Differential Equations

Subject Teacher
Santosh Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

ssdhamone@acscollegeonde.ac.in

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Contents

Compatible Differential Equations

Compatible Differential Equations

Let $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ be first order partial differentiable equations.If every solution of $f = 0$ is also solution of $g = 0$ and

Jacobian $J = \dfrac{\partial(f, g)}{\partial(p, q)} \neq 0$

then this two equations f and g are said to be Compatiable.

Compatible Differential Equations

Theorem :

Show that the condition for $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ compatiable is $[f, g] = 0$
i.e. $\frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$

Proof

Let

$$f(x,\ y,\ z,\ p,\ q) = 0 \tag{1}$$

$$g(x,\ y,\ z,\ p,\ q) = 0 \tag{2}$$

Compatible Differential Equations

Proof of Theorem Continue...

be first order partial differential equations. From (1) and (2) we obtain

$$p = \Phi(x, y, z), \quad q = \Psi(x, y, z)$$

The condition that equations (1) and (2) should be compatible reduces to $p\ dx + q\ dy = dz$ is integrable.

$$\therefore \quad \Phi\ dx + \Psi\ dy - dz = 0 \tag{3}$$

is integrable.

Let $\bar{X} = (\Phi, \Psi, -1)$ then $\bar{X}.Curl\bar{X} = 0$

Compatible Differential Equations

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Proof of Theorem Continue...

$$\text{Now, } Curl\bar{X} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \Phi & \Psi & -1 \end{vmatrix}$$

$$= (0 - \frac{\partial \Psi}{\partial z})\hat{i} - (0 - \frac{\partial \Phi}{\partial z})\hat{j} + (\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y})\hat{k}$$

$$= -\frac{\partial \Psi}{\partial z}\hat{i} + \frac{\partial \Phi}{\partial z}\hat{j} + (\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y})\hat{k}$$

$$\bar{X}.Curl\bar{X} = (\Phi\hat{i}, \Psi\hat{j}, -1\hat{k}).[-\frac{\partial \Psi}{\partial z}\hat{i} + \frac{\partial \Phi}{\partial z}\hat{j} + (\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y})\hat{k}]$$

$$-\Phi\frac{\partial \Psi}{\partial z} + \Psi\frac{\partial \Phi}{\partial z} - \frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y} = 0$$

Compatible Differential Equations

Proof of Theorem Continue...

$$\Psi \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial y} = \Phi \frac{\partial \Psi}{\partial z} + \frac{\partial \Psi}{\partial x} \qquad (4)$$

Differentiate (1) w.r.t x and z,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} . \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} . \frac{\partial q}{\partial x} = 0$$

$$\therefore \quad f_x + f_p \Phi_x + f_q \Psi_x = 0 \qquad (5)$$

$$\text{and } \frac{\partial f}{\partial z} + \frac{\partial f}{\partial p} . \frac{\partial p}{\partial z} + \frac{\partial f}{\partial q} . \frac{\partial q}{\partial z} = 0$$

Compatible Differential Equations

Proof of Theorem Continue...

$$\therefore \quad f_z + f_p\Phi_z + f_q\Psi_z = 0 \tag{6}$$

Multiply equation (6) by Φ then add it to equation (5)

$$(f_x + \Phi f_z) + f_p(\Phi_x + \Phi\Phi_z) + f_q(\Psi_x + \Phi\Psi_z) = 0 \tag{7}$$

Differentiate (2) w.r.t x and z and as above, we get,

$$(g_x + \Phi g_z) + g_p(\Phi_x + \Phi\Phi_z) + g_q(\Psi_x + \Phi\Psi_z) = 0 \tag{8}$$

Compatible Differential Equations

Proof of Theorem Continue...

Multiply equation (7) by g_p and (8) by f_p then take (7)-(8)

$$g_p(f_x+\Phi f_z)+g_pf_p(\Phi_x+\Phi\Phi_z)+g_pf_q(\Psi_x+\Phi\Psi_z)-f_p(g_x+\Phi g_z)-f_pg_p(\Phi_x+\Phi\Phi_z)-f_pg_q(\Psi_x+\Phi\Psi_z)=0$$

$$g_p(f_x+\Phi f_z)+g_pf_q(\Psi_x+\Phi\Psi_z)-f_p(g_x+\Phi g_z)-f_pg_q(\Psi_x+\Phi\Psi_z)=0$$

$$g_p(f_x+\Phi f_z)-f_p(g_x+\Phi g_z)+(\Psi_x+\Phi\Psi_z)(g_pf_q-f_pg_q)=0$$

$$\Phi(g_pf_z-f_pg_z)+(f_xg_p-g_xf_p)+(\Psi_x+\Phi\Psi_z)(g_pf_q-f_pg_q)=0$$

$$(f_xg_p-g_xf_p)+\Phi(g_pf_z-f_pg_z)=(\Psi_x+\Phi\Psi_z)(f_pg_q-(g_pf_q)$$

$$\therefore\quad \frac{\partial(f,g)}{\partial(x,p)}+\Phi\frac{\partial(f,g)}{\partial(z,p)}=J(\Psi_x+\Phi\Psi_z)$$

Compatible Differential Equations

Proof of Theorem Continue...

$$\therefore \quad (\Psi_x + \Phi\Psi_z) = \frac{1}{J} \quad \left[\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)}\right] \qquad (9)$$

Similarly diff. eq^{ns} (1) and (2) w.r.t y and z, we obtain

$$\therefore \quad (\Phi_y + \Psi\Phi_z) = \frac{-1}{J} \quad \left[\frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}\right] \qquad (10)$$

Using equation (4) we get,

$$\frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q} = 0$$

$[f,g] = 0$ It is condition for f and g are to be compatible.

Compatible Differential Equations

Problem 1:

Show that the PDE $xp = yq$ and $z(xp + yq) = 2xy$ are compatible. Find Solution

Solution:

Let

$$f = x\ p - y\ q = 0 \tag{11}$$

$$g = z(xp + yq) - 2xy = 0 \tag{12}$$

Compatible Differential Equations

Solution of Problem 1 Continue...

$$\frac{\partial(f,g)}{\partial(x,p)}=\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial p}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p}\end{vmatrix}=\frac{\partial f}{\partial x}\ \frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\ \frac{\partial g}{\partial x}$$

$$= p(zx) - x(zp - 2y) = zpx - zpx + 2xy = 2xy$$

$$\frac{\partial(f,g)}{\partial(x,p)} = 2xy$$

$$\frac{\partial(f,g)}{\partial(y,q)}=\begin{vmatrix}\frac{\partial f}{\partial y} & \frac{\partial f}{\partial q}\\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q}\end{vmatrix} = \begin{vmatrix}-q & -y\\ zq-2x & zy\end{vmatrix} = -2xy$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -2xy$$

Compatible Differential Equations

Solution of Problem 1 Continue...

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \dfrac{\partial f}{\partial z} & \dfrac{\partial f}{\partial p} \\ \dfrac{\partial g}{\partial z} & \dfrac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp + yq & zx \end{vmatrix} = -x(xp + yq)$$

$$\frac{\partial(f,g)}{\partial(z,p)} = -x(xp + yq)$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \dfrac{\partial f}{\partial z} & \dfrac{\partial f}{\partial q} \\ \dfrac{\partial g}{\partial z} & \dfrac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp + yq & zy \end{vmatrix} = y(xp + yq)$$

$$\frac{\partial(f,g)}{\partial(z,q)} = y(xp + yq)$$

Compatible Differential Equations

Solution of Problem 1 Continue...

Condition for Compatible is

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q}$$

$$= 2xy - 2xy + p[-x(xp+yq)] + qy(xp+yq)$$

$$=-x^2p^2 - xypq + xypq + y^2q^2$$

$$=y^2q^2 - x^2p^2$$

$$[f,g] = 0$$

$\therefore$ f and g satisfies the condition of Compatibility.

$\therefore$ Given PDE are compatible.

Solution of Problem 1 Continue...

By equation (11) $xp = yq$

Using this in (12) $z(xp + xp) = 2xy$

$2xpz = 2xy \implies p = \dfrac{y}{z}$

Using value of p in (11), we get

$x(\dfrac{y}{z}) = yq \implies q = \dfrac{x}{z}$

Using p and q in $p\ dx + q\ dy = dz$

$\therefore \dfrac{y}{z}dx + \dfrac{x}{z}dy = dz \implies ydx + x\,dy = zdz$

$\therefore d(xy) = zdz$

Integrating, we get

$xy = \dfrac{z^2}{2} + c$

$2xy - z^2 = c$ .... Required Solution.

Compatible Differential Equations

Problem 2:

Show that the PDE $xp - yq = x$ and $x^2p + q = xz$ are compatible. Hence find Solution

Solution:

Let

$$f = x\ p\ \text{-}\ y\ q\ \text{-}\ x = 0 \tag{13}$$

$$g = x^2 p + q - xz = 0 \tag{14}$$

Compatible Differential Equations

Solution of Problem 2 Continue...

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix} = x^2(p-1) - x(2xp-z)$$

$$\frac{\partial(f,g)}{\partial(x,p)} = x^2(p-1) - x(2xp-z)$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix} = -q$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -q$$

Compatible Differential Equations

Solution of Problem 2 Continue...

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix} = x^2$$

$$\frac{\partial(f,g)}{\partial(z,p)} = x^2$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix} = -xy$$

$$\frac{\partial(f,g)}{\partial(z,q)} = -xy$$

Compatible Differential Equations

Solution of Problem 2 Continue...

Condition for Compatible is

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q}$$

$$= x^2(p-1) - x(2xz - z) + px^2 - q - qxy$$

$$= \text{-}x^2p - x^2 - 2x^2p + xz + x^2p - q - qxy$$

$$= (\text{xz - q) - x}^2 - qxy$$

$$= x^2p - x^2 - qxy \text{ .... by equation (14)}$$

$$= \text{x(xp - yq) -x}^2$$

$$= \text{x.x - x}^2$$

$$[f,g] = 0$$

$\therefore$ f and g satisfies the condition of Compatibility.

$\therefore$ Given PDE are compatible.

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Solution of Problem 2 Continue...

Multiply equation (14) by y then add it in equation (13)

$$(x + x^2y)\,p = x + xyz \implies x(1 + xy)\quad p = x\,(1 + yz)$$

$$p = \frac{1 + yz}{1 + xy}$$

Using it in (13)

$$\frac{1 + yz}{1 + xy} - yq = x \implies \frac{1 + yz}{1 + xy} - x = yq$$

$$\implies yq = \frac{x + xyz - x - x^2y}{1 + xy} \implies yq = \frac{y(xz - x^2)}{1 + xy}$$

$$q = \frac{x(z - x)}{1 + xy}$$

Compatible Differential Equations

Solution of Problem 2 Continue...

Using p and q in $p\ dx + q\ dy = dz$

$$\therefore \quad \frac{1+yz}{1+xy}dx + \frac{x(z-x)}{1+xy}dy = dz$$

It is Pfaffian Differential Equation

Take x = constant $\implies dx = 0$

$\therefore$ (xz - x^2)$dy - (1+xy)dz = 0$

$\therefore$ x(z - x) dy - (1 + xy) dz = 0

Dividing throughout by $(z-x)(1+xy)$

$$\therefore \quad \frac{x}{1+xy}dy - \frac{dz}{z-x} = 0$$

Integrating, we get

$$\ln(1+xy) - \ln(z-x) = \ln c_1$$

Compatible Differential Equations

Solution of Problem 2 Continue...

$$\therefore \quad \frac{1+xy}{z-x} = c_1$$ Hence Solution is of the form

$$\frac{1+xy}{z-x} = \Phi(x)$$

Hence Required Solution is

$$\frac{1+xy}{z-x} = c$$

Charpit's Method

Charpit's Auxiliary Equation

Show that the Charpit's Auxiliary Equation is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

Proof

Let

$$f(x, y, z, p, q) = 0 \tag{15}$$

be first order partial differential equation where x , y are independent and z is dependent variable.

Charpit's Auxiliary Equation

Proof of Charpit's Auxiliary Equation...

$$\therefore z = z\,(x\,,\,y)$$
$$\therefore dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\therefore \quad dz = p \quad dx + q \quad dy \tag{16}$$

In this method we consider a another relation

$$F(x, y, z, p, q) = 0 \tag{17}$$

such that values of p and q obtained from (15) and (17) makes equation (16) integrable.

Charpit's Auxiliary Equation

Proof of Charpit's Auxiliary Equation...

The solution of (16) is complete integral of (15).

Differentiate (15) and (17) w.r.t. x

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0 \tag{18}$$

Similarly,

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + \frac{\partial F}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}.\frac{\partial q}{\partial x} = 0 \tag{19}$$

Charpit's Auxiliary Equation

Proof of Charpit's Auxiliary Equation...

Multiply to (18) by $\dfrac{\partial F}{\partial p}$ and (19) by $\dfrac{\partial f}{\partial p}$ then take (18) -(19)

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0$$

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + p\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right) + \frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right) = 0 \tag{20}$$

Charpit's Auxiliary Equation

Proof of Charpit's Auxiliary Equation...

Differentiate (15) and (17) w.r.t. y, we get
[In (20) replace x = y , p = q , q = p]

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)+\frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)=0 \tag{21}$$

$$\text{Now,}\frac{\partial q}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial y\partial x}$$

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial p}{\partial y}\implies\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y}$$

Charpit's Auxiliary Equation

Proof of Charpit's Auxiliary Equation...

Adding equations (20) and (21), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)+$$

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)=0$$

$$\therefore \frac{\partial F}{\partial x}(-\frac{\partial f}{\partial p})+\frac{\partial F}{\partial y}(-\frac{\partial f}{\partial q})+\frac{\partial F}{\partial z}(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q})+$$

$$\frac{\partial F}{\partial p}(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z})+\frac{\partial F}{\partial q}(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z})=0$$

Charpit's Auxiliary Equation

Proof of Charpit's Auxiliary Equation...

It's Auxiliary Equation is

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z}$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

It is known as Charpit's Auxiliary equation.
Finding expression for p and q from (15) and Charpit's Auxiliary equation putting this value in (16) and on integration, we get required result.

Solution of PDE by using Charpit's Method

Problem 1:

Solve the PDE $(p^2+q^2)y = qz$ by Charpit's method

Solution

Let

$$f = (p^2+q^2)y - qz = 0 \tag{22}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$

$$\frac{\partial f}{\partial y} = f_y = p^2+q^2$$

Solution of PDE by using Charpit's Method

Solution of Problem 1 Continue...

$$\frac{\partial f}{\partial z} = f_z = -q$$

$$\frac{\partial f}{\partial p} = f_p = 2py$$

$$\frac{\partial f}{\partial q} = f_q = 2qy - z$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + 2q^2y - qz} = \frac{-dp}{0 + (-pq)} = \frac{-dq}{p^2 + q^2 - q^2}$$

Solution of PDE by using Charpit's Method

Solution of Problem 1 Continue...

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + 2q^2y - qz} = \frac{-dp}{-pq} = \frac{-dq}{p^2}$$

Consider two last ratios

$$\frac{dp}{pq} = \frac{-dq}{p^2} \implies \frac{dp}{q} = \frac{-dq}{p} \implies \text{pdp} = \text{-qdq}$$

Integrating,

$$\frac{p^2}{2} + \frac{q^2}{2} = a \implies \text{p}^2 + q^2 = a$$

Using $p^2 + q^2 = a$ in equation (22)

$$ay = qz \implies q = \frac{ay}{z}$$

Using $q = \frac{ay}{z}$ in $p^2 + q^2 = a$

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Solution of Problem 1 Continue...

$$p^2 + (\frac{ay}{z})^2 = a \implies p^2 = a - \frac{a^2y^2}{z^2}$$

$$\implies p^2 = \frac{az^2 - a^2y^2}{z^2} \implies p = \frac{\sqrt{az^2 - a^2y^2}}{z}$$

Consider, $p\ dx + q\ dy = dz$

$$\therefore \frac{\sqrt{az^2 - a^2y^2}}{z}\ dx + \frac{ay}{z}dy = dz$$

$$\sqrt{az^2 - a^2y^2}\ dx + \text{ay dy} = \text{z dz}$$

$$\sqrt{a}(\sqrt{z^2 - ay^2})\ dx = \text{z dz - ay dy}$$

$$\sqrt{a}\ dx = \frac{zdz - aydy}{\sqrt{z^2 - ay^2}}$$

Integrating we get,

$\sqrt{a}\ x = \sqrt{z^2 - ay^2} + b$ ...Required Solution.

Solution of PDE by using Charpit's Method

Problem 2:

Solve the PDE $p = (z + qy)^2$ by Charpit's method

Solution

Let

$$f = p - (z + qy)^2 = 0 \tag{23}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$

$$\frac{\partial f}{\partial y} = f_y = -2(z + qy).q$$

Solution of PDE by using Charpit's Method

Solution of Problem 2 Continue...

$$\frac{\partial f}{\partial z} = f_z = -2(z + qy)$$

$$\frac{\partial f}{\partial p} = f_p = 1$$

$$\frac{\partial f}{\partial q} = f_q = -2(z + qy).y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{1} = \frac{dy}{-2y(z + qy)} = \frac{dz}{p - 2yq(z + qy)} =$$

$$\frac{-dp}{-2p(z + qy)} = \frac{-dq}{-2q(z + qy) - 2q(z + qy)}$$

Solution of PDE by using Charpit's Method

Solution of Problem 2 Continue...

$$\text{Consider} \quad \frac{dy}{-2y(z+qy)} = \frac{-dp}{-2p(z+qy)}$$

$$\therefore \frac{dy}{-y} = \frac{dp}{p}$$

Integrating,

$$-\ln y = \ln p - \ln a \implies \ln y + \ln p = \ln a \implies \text{yp} = \text{a}$$

$$\therefore p = \frac{a}{y}$$

$$\text{Using } p = \frac{a}{y} \text{ in equation (23)}$$

$$\frac{a}{y} = (z+qy)^2 \implies \text{z} + \text{qy} = \sqrt{\frac{a}{y}} \implies \text{qy} = \sqrt{\frac{a}{y}} - z$$

Solution of PDE by using Charpit's Method

Solution of Problem 2 Continue...

$$\therefore\ q = \frac{1}{y}\left(\sqrt{\frac{a}{y}} - z\right)$$

Consider, $p\ dx + q\ dy = dz$

$$\therefore\ \frac{a}{y}\ dx + \ \frac{1}{y}\left(\sqrt{\frac{a}{y}} - z\right) dy = dz$$

$$adx + \sqrt{\frac{a}{y}}dy = ydz + zdy$$

$$a\ \ dx + \sqrt{\frac{a}{y}}dy = d(yz)$$

Integrating we get,

$$a\,x + 2\sqrt{ay} = yz + b$$

$$ax + 2\sqrt{ay} - yz = b$$

This is Required Complete Integral.

Solution of PDE by using Charpit's Method

Problem 3:

Solve the PDE $2xz - px^2 - 2qxy + pq = 0$ by Charpit's method

Solution

Let

$$f = 2xz - px^2 - 2qxy + pq = 0 \tag{24}$$

$$\frac{\partial f}{\partial x} = f_x = 2z - 2px - 2qy$$

$$\frac{\partial f}{\partial y} = f_y = -2qx$$

Solution of PDE by using Charpit's Method

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Santosh Dhamone

Solution of Problem 3 Continue...

$$\frac{\partial f}{\partial z} = f_z = 2x$$

$$\frac{\partial f}{\partial p} = f_p = x^2 + q$$

$$\frac{\partial f}{\partial q} = f_q = p - 2xy$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p f_p + q f_q} = \frac{-dp}{f_x + p f_z} = \frac{-dq}{f_y + q f_z}$$

$$\frac{dx}{x^2+q} = \frac{dy}{p-2xy} = \frac{dz}{p(x^2+q)+q(p-2xy)} = \frac{-dp}{2(z-px-qy)+2px} = \frac{-dq}{-2qx+2qx}$$

$$\frac{dx}{x^2+q} = \frac{dy}{p-2xy} = \frac{dz}{px^2+2pq-2qxy} = \frac{-dp}{2z-2qy} = \frac{dq}{0}$$

Solution of PDE by using Charpit's Method

Solution of Problem 3 Continue...

Consider,

$$\text{Each Ratio} = \frac{dq}{0}$$

$$\therefore \text{dq} = 0$$

Integrating,we get

$$\therefore \quad q = a$$

Using $q = a$ in equation (24)

$$2xz - px^2 - 2axy + pa = 0$$

$$\therefore \text{p(a - x}^2) = 2axy - 2xz$$

$$\therefore \quad \text{p} = \frac{2x(ay - z)}{a - x^2}$$

Solution of PDE by using Charpit's Method

Solution of Problem 3 Continue...

Consider, $p \ dx + q \ dy = dz$

$$\therefore \ \frac{2x(ay-z)}{a-x^2} \ dx + \text{a} \quad \text{dy} = \text{dz}$$

Divide throughout by $(ay - z)$, we get,

$$\therefore \ \frac{2x}{a-x^2} \ dx + \frac{a}{(ay-z)} \ dy = \frac{dz}{(ay-z)}$$

$$\therefore \text{-} \ \frac{-2x}{a-x^2} \ dx + \frac{ady - dz}{(ay-z)} = 0$$

Integrating we get,

$$-\ln\left(a - x^2\right) + \ln\left(ay - z\right) = \ln b$$

$$\ln \frac{ay-z}{a-x^2} = \ln b$$

$\dfrac{ay-z}{a-x^2} = b$.... This is Required Complete Integral.

Solution of PDE by using Charpit's Method

Problem 4:

Solve the PDE $2(z+xp+yq) = yp^2$ by Charpit's method

Solution

Let

$$f = 2(z + xp + yq) - yp^2 = 0 \tag{25}$$

$$\frac{\partial f}{\partial x} = f_x = 2p$$

$$\frac{\partial f}{\partial y} = f_y = 2q - p^2$$

Solution of PDE by using Charpit's Method

Solution of Problem 4 Continue...

$$\frac{\partial f}{\partial z} = f_z = 2$$
$$\frac{\partial f}{\partial p} = f_p = 2x - 2yp$$
$$\frac{\partial f}{\partial q} = f_q = 2y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$
$$\frac{dx}{2x - 2yp} = \frac{dy}{2y} = \frac{dz}{2xp - 2yp^2 + 2yq} = \frac{-dp}{2p + 2p} = \frac{-dq}{2q - p^2 + 2q}$$
$$\frac{dx}{2x - 2yp} = \frac{dy}{2y} = \frac{dz}{2xp - 2yp^2 + 2yq} = \frac{-dp}{4p} = \frac{-dq}{4q - p^2}$$

Solution of Problem 4 Continue...

Consider,

$$\frac{dy}{2y} = \frac{-dp}{4p}$$

Integrating,we get

$$2\ln y = -\ln p + \ln a$$

$$\ln y^2 + \ln p = \ln a \implies \ln y^2 p = \ln a$$

$$\therefore \quad p = \frac{a}{y^2}$$

Using $p = \dfrac{a}{y^2}$ in equation (25)

$$2z + 2x\frac{a}{y^2} + 2yq - y(\frac{a}{y^2})^2 = 0$$

$$\therefore \quad 2z + \frac{2ax}{y^2} + 2yq - (\frac{a^2}{y^3}) = 0$$

Solution of PDE by using Charpit's Method

Solution of Problem 4 Continue...

$$\therefore \quad q = \frac{a^2}{2y^4} - \frac{z}{y} - \frac{xa}{y^3}$$

Consider, $p \ dx + q \ dy = dz$

$$\therefore \quad \frac{a}{y^2} \ dx + \left(\frac{a^2}{2y^4} - \frac{z}{y} - \frac{xa}{y^3} \right) \ dy = dz$$

$$\therefore \quad \frac{a}{y^2} \ dx + \frac{a^2}{2y^4} dy - \frac{z}{y} dy - \frac{xa}{y^3} dy = dz$$

$$a \left(\frac{ydx - xdy}{y^3} \right) + \frac{a^2}{2} \frac{dy}{y^3} = ydz + zdy$$

$$ad(\frac{x}{y}) + \frac{a^2}{2} \frac{dy}{y^3} = d(yz)$$

Integrating we get,

Solution of PDE by using Charpit's Method

Solution of Problem 4 Continue...

Integrating we get,

$$a\frac{x}{y} - \frac{a^2}{4y^2} = yz + b$$

$$\frac{ax}{y} - \frac{a^2}{4y^2} - yz = b$$

This is Required Complete Integral.

Solution of PDE by using Charpit's Method

Problem 5:

Solve the PDE $pxy + pq + qy = yz$ by Charpit's method

Solution

Let

$$f = \text{pxy} + \text{pq} + \text{qy - yz} = 0 \tag{26}$$

$$\frac{\partial f}{\partial x} = f_x = py$$
$$\frac{\partial f}{\partial y} = f_y = q - z$$

Solution of PDE by using Charpit's Method

Solution of Problem 5 Continue...

$$\frac{\partial f}{\partial z} = f_z = -y$$
$$\frac{\partial f}{\partial p} = f_p = xy + q$$
$$\frac{\partial f}{\partial q} = f_q = p + y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$
$$\frac{dx}{xy+q} = \frac{dy}{p+y} = \frac{dz}{xyp + qp + pq + yq} = \frac{-dp}{py - yp} = \frac{-dq}{q - z - qy}$$
$$\frac{dx}{xy+q} = \frac{dy}{p+y} = \frac{dz}{xyp + 2pq + yq} = \frac{-dp}{0} = \frac{-dq}{q - z - qy}$$

Solution of PDE by using Charpit's Method

Solution of Problem 5 Continue...

Consider,

$$EachRatio = \frac{dp}{0}$$

$$\therefore \text{dp} = 0$$

Integrating,we get

$$\therefore \quad p = a$$

Using $p = a$ in equation (26)

$$axy + aq + qy - yz = 0$$

$$q(a + y) = yz - axy$$

$$\therefore \quad q = \frac{y(z - ax)}{a + y}$$

Solution of PDE by using Charpit's Method

Solution of Problem 5 Continue...

$$\text{Consider, } p\ dx + q\ dy = dz$$

$$\therefore\quad a\ \ \mathrm{dx} + \frac{y(z-ax)}{a+y}\ dy = dz$$

$$\therefore\quad \frac{y(z-ax)}{a+y}\ dy = dz - a\ dx$$

$$\therefore\quad \frac{y}{a+y}\ dy = \frac{dz - adx}{(z-ax)}$$

$$\therefore\quad \frac{y+a-a}{a+y}\ dy = \frac{dz - adx}{(z-ax)}$$

$$\therefore\quad \left(1 - \frac{a}{a+y}\right)\ dy = \frac{dz - adx}{(z-ax)}$$

Solution of PDE by using Charpit's Method

Solution of Problem 5 Continue...

Integrating we get,

$$y - a\ln(a+y) = \ln(z-ax) + b$$
$$y - \ln(a+y)^a - \ln(z-ax) = b$$
$$y - [\ln(a+y)^a + \ln(z-ax)] = b$$
$$y - \ln(a+y)^a(z-ax) = b$$
$$y - b = \ln(a+y)^a(z-ax)$$
$$(a+y)^a(z-ax) = e^{y-b}$$
$$(a+y)^a(z-ax) = ce^y$$

This is Required Complete Integral.

Solution of PDE by using Charpit's Method

Problem 6:

Solve the PDE $p^2x + q^2y = z$ by Charpit's method

Solution

Let

$$f = p^2x + q^2y - z = 0 \tag{27}$$

$$\frac{\partial f}{\partial x} = f_x = p^2$$
$$\frac{\partial f}{\partial y} = f_y = q^2$$

Solution of PDE by using Charpit's Method

Solution of Problem 6 Continue...

$$\frac{\partial f}{\partial z} = f_z = -1$$

$$\frac{\partial f}{\partial p} = f_p = 2px$$

$$\frac{\partial f}{\partial q} = f_q = 2qy$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2p^2x + 2q^2y} = \frac{-dp}{p^2 - p} = \frac{-dq}{q^2 - q}$$

Solution of PDE by using Charpit's Method

Solution of Problem 6 Continue...

Consider,

$$\text{Each Ratio} = \frac{p^2dx + 2pxdp}{2p^2x}$$

Similarly,

$$\text{Each Ratio} = \frac{q^2dy + 2qydq}{2q^2y}$$

$$\frac{p^2dx + 2pxdp}{2p^2x} = \frac{q^2dy + 2qydq}{2q^2y}$$

Integrating,we get

$$\ln p^2x = \ln q^2y + \ln a$$

$$\ln p^2x = \ln aq^2y \implies \mathrm{p}^2x = aq^2y \implies \mathrm{p}^2 = \frac{ayq^2}{x}$$

$$\therefore\ p = \sqrt{\frac{ay}{x}}q$$

Solution of PDE by using Charpit's Method

Solution of Problem 6 Continue...

Using $p = \sqrt{\frac{ay}{x}}q$ in equation (27)

$$\frac{ayq^2}{x}x + q^2y = z$$

$$q^2y(1+a) = z \implies \mathrm{q}^2 = \frac{z}{y(1+a)}$$

$$\therefore \quad q = \sqrt{\frac{z}{y(1+a)}}$$

using value of q in p

$$p = \sqrt{\frac{ay}{x}}\sqrt{\frac{z}{y(1+a)}}$$

$$\therefore \quad p = \sqrt{\frac{az}{x(1+a)}}$$

Solution of PDE by using Charpit's Method

Solution of Problem 6 Continue...

Consider, $p\ dx + q\ dy = dz$

$$\sqrt{\frac{az}{x(1+a)}}dx + \sqrt{\frac{z}{y(1+a)}}dy = dz$$

$$\sqrt{\frac{a}{(1+a)}}\frac{dx}{\sqrt{x}} + \sqrt{\frac{1}{(1+a)}}\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Integrating we get,

$$\sqrt{\frac{a}{(1+a)}}2\sqrt{x} + \sqrt{\frac{1}{(1+a)}}2\sqrt{y} = 2\sqrt{z} + b$$

$$\sqrt{ax} + \sqrt{y} = \sqrt{a+1}(\sqrt{z} + b)$$

This is Required Complete Integral.

Solution of PDE by using Charpit's Method

Problem 7:

Solve the PDE $p^2x + qy = z$ by Charpit's method

Solution

Let

$$f = p^2x + qy - z = 0 \tag{28}$$

$$\frac{\partial f}{\partial x} = f_x = p^2$$
$$\frac{\partial f}{\partial y} = f_y = q$$

Solution of PDE by using Charpit's Method

Solution of Problem 6 Continue...

$$\frac{\partial f}{\partial z} = f_z = -1$$

$$\frac{\partial f}{\partial p} = f_p = 2px$$

$$\frac{\partial f}{\partial q} = f_q = y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{2px} = \frac{dy}{y} = \frac{dz}{2p^2x + qy} = \frac{-dp}{p^2 - p} = \frac{-dq}{q - q}$$

Solution of PDE by using Charpit's Method

Solution of Problem 7 Continue...

Consider,

$$EachRatio = \frac{dq}{0}$$

$$\therefore \ \mathrm{dq} = 0$$

Integrating,we get

$$\therefore \ \ q = a$$

Using $q = a$ in equation (28)

$$p^2x + ay - z = 0 \implies \mathrm{p}^2x = z - ay \implies \mathrm{p}^2 = \frac{z - ay}{x}$$

$$p = \sqrt{\frac{z - ay}{x}}$$

$$\therefore \ \ p = \sqrt{\frac{z - ay}{x}}$$

Solution of PDE by using Charpit's Method

Solution of Problem 7 Continue...

Consider, $p \; dx + q \; dy = dz$

$$\sqrt{\frac{z - ay}{x}} dx + ady = dz$$

$$\sqrt{z - ay} \frac{dx}{\sqrt{x}} = dz - ady$$

$$\frac{dx}{\sqrt{x}} = \frac{dz - ady}{\sqrt{z - ay}}$$

Integrating we get,

$$2\sqrt{x} = 2\sqrt{z - ay} + b$$

$$\sqrt{x} - \sqrt{z - ay} = b$$

This is Required Complete Integral.

Special types of First Order Partial Differential Equations

Type 1: PDE involving p and q only

PDE involving p and q only

The equation containing p and q only is of the form

$$f(\ p\ ,\ q\ ) = 0 \tag{29}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$

$$\frac{\partial f}{\partial y} = f_y = 0$$

Special types of First Order Partial Differential Equations

Type 1: PDE involving p and q only Continue...

$$\frac{\partial f}{\partial z} = f_z = 0$$

$$\frac{\partial f}{\partial p} = f_p$$

$$\frac{\partial f}{\partial q} = f_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{0} = \frac{-dq}{0}$$

Special types of First Order Partial Differential Equations

Type 1: PDE involving p and q only Continue...

$$\text{Now, } EachRatio = \frac{-dp}{0}$$

$$\therefore \text{ dp} = 0$$

$$\text{Integrating we get ,} \quad p = a$$

$$\text{Using } p = a \text{ in equation (29), we obtain}$$

$$q = \Phi(a)$$

$$\text{using values of p and q in } p\ dx + q\ dy = dz$$

$$a\ dx + \Phi(a)\ dy = dz$$

$$\text{Integrating}$$

$$ax + \Phi(a)y = z + b$$

$$ax + \Phi(a)y - z = b$$

Type 1: PDE involving p and q only

Problem 1:

Solve the PDE $p + q = pq$

Solution

Let

$$f = p + q - pq = 0 \tag{30}$$

It contains only p and q.

$\therefore$ p = a = constant

$\therefore \; p = a = constant$

Using $p = a$ in equation (30)

$\therefore$ a + q = aq

Type 1: PDE involving p and q only

Solution of Problem 1 Continue...

$$\therefore aq - q = a$$

$$\therefore q(a - 1) = a$$

$$\therefore q = \frac{a}{a-1}$$

$$q = \frac{a}{a-1}$$

Consider, $p \; dx + q \; dy = dz$

$$a \; dx + \frac{a}{a-1} \; dy = dz$$

Integrating we get, $ax + \dfrac{ay}{a-1} = z$

$a(a-1)x + ay = (a-1)z$... Required Solution

Special types of First Order Partial Differential Equations

Type 2: PDE not involving independent variables x and y

Type 2:PDE not involving independent variables x and y

The equation containing p and q only is of the form

$$f(\ p\ ,\ q\ ,\ z\ ) = 0 \tag{31}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$
$$\frac{\partial f}{\partial y} = f_y = 0$$

Special types of First Order Partial Differential Equations

Type 2: PDE not involving independent variables x and y

$$\frac{\partial f}{\partial z} = f_z$$

$$\frac{\partial f}{\partial p} = f_p$$

$$\frac{\partial f}{\partial q} = f_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{pf_z} = \frac{-dq}{qf_z}$$

Special types of First Order Partial Differential Equations

Type 2: PDE not involving independent variables x and y Continue

$$\implies \frac{-dp}{pf_z} = \frac{-dq}{qf_z}$$

$$\implies \frac{-dp}{p} = \frac{-dq}{q}$$

Integrating we get,

$$\ln p = \ln q + \ln a$$

$$\ln p = \ln aq$$

$$p = aq$$

Using $p = a$ in equation (31), we obtain expression for q using values of p and q in $p\ dx + q\ dy = dz$

Integrating we get required solution

Type 2: PDE not involving independent variables

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Problem 1:

Solve the PDE $p^2z^2 + q^2 = 1$

Solution

It is of the form

$$f(p, q, z) = p^2z^2 + q^2 - 1 = 0 \qquad (32)$$

$$\text{Put} p = aq \text{ in (32) then } a^2q^2z^2 + q^2 = 1$$

$$\implies q^2(a^2z^2 + 1) = 1 \implies q^2 = \frac{1}{a^2z^2 + 1}$$

$$q = \frac{1}{\sqrt{a^2z^2 + 1}}$$

Type 2: PDE not involving independent variables

Solution of Problem 1 Continue...

Using value of q in $p = aq$

$$p = \frac{a}{\sqrt{a^2z^2+1}}$$

Consider, $p\ dx + q\ dy = dz$

$$\frac{a}{\sqrt{a^2z^2+1}}\ dx + \frac{1}{\sqrt{a^2z^2+1}}\ dy = dz$$

$$adx + dy = \sqrt{a^2z^2+1}dz$$

Integrating we get,

$$ax + y = a\int\sqrt{z^2+\frac{1}{a^2}}$$

$$ax + y = a\left[\frac{z}{z}\sqrt{z^2+\frac{1}{a^2}} + \frac{1}{2a^2}\ln\left(z + +\sqrt{z^2+\frac{1}{a^2}}\right)\right]$$

This is Required Solution

Type 2: PDE not involving independent variables

Problem 2:

Solve the PDE $pq + q^3 = 3pzq$

Solution

It is of the form

$$f(p, q, z) = pq + q^3 - 3pzq = 0 \tag{33}$$

$$\text{Put } p = aq \text{ in (33) then } aq^2 + q^3 = 3aq^2 z$$
$$\implies q^2(a + q) = 3aq^2 z$$
$$\implies (a + q) = 3az$$
$$q = 3az - a = a(3z - 1)$$

Type 2: PDE not involving independent variables

Solution of Problem 2 Continue...

Using value of q in $p = aq$

$$p = a^2(3z - 1)$$

Consider, $p\ \ dx + q\ \ dy = dz$

$$a^2(3z - 1)\ \ dx + a(3z - 1)\ \ dy = dz$$

$$a^2 dx + ady = \frac{dz}{3z - 1}$$

Integrating we get,

$$a^2 x + ay = \frac{\ln(3z - 1)}{3} + b$$

$$a^2 x + ay = \frac{\ln(3z - 1)}{3} + b$$

This is Required Solution

Type 2: PDE not involving independent variables

Problem 3:

Find the complete integral of $z^2(p^2z^2 + q^2) = 1$

Solution

It is of the form

$$f(p, q, z) = z^2(p^2z^2 + q^2) - 1 = 0 \tag{34}$$

Put $p = aq$ in (34) then $z^2(a^2q^2z^2 + q^2) = 1$

$$\implies z^2q^2(a^2z^2 + 1) = 1 \implies q^2 = \frac{1}{z^2(a^2z^2 + 1)}$$

$$q = \frac{1}{z\sqrt{a^2z^2 + 1}}$$

Type 2: PDE not involving independent variables

Solution of Problem 3 Continue...

Using value of q in $p = aq$

$$p = \frac{a}{z\sqrt{a^2z^2+1}}$$

Consider, $p\ dx + q\ dy = dz$

$$\frac{a}{z\sqrt{a^2z^2+1}}\ dx + \frac{1}{z\sqrt{a^2z^2+1}}\ dy = dz$$

$$adx + dy = z\sqrt{1+a^2z^2}dz$$

Integrating we get,

$$ax + y = \int z\sqrt{1+a^2z^2}dz + b$$

Type 2: PDE not involving independent variables

Solution of Problem 3 Continue...

For integration, we use substitution method

$$\text{Put } 1 + a^2z^2 = t$$

$$\Longrightarrow 2\mathrm{a}^2 zdz = dt$$

$$\Longrightarrow \mathrm{z\ dz} = \frac{dt}{2a^2}$$

$$ax + y - b = \int \sqrt{t}\frac{dt}{2a^2}$$

$$ax + y - b = \frac{1}{2a^2}\frac{t^{\frac{3}{2}}}{\frac{3}{2}}$$

$$3a^2(ax + y - b) = (a^2z^2 + 1)^{\frac{3}{2}}$$

This is Required Solution.

Type 2: PDE not involving independent variables

Problem 4:

Find the complete integral of $q^2 = z^2p^2(1 - p^2)$

Solution

It is of the form

$$f(p, q, z) = q^2 - z^2p^2(1 - p^2) = 0 \qquad (35)$$

Put $p = aq$ in (34) then $q^2 = z^2a^2q^2(1 - a^2q^2)$

$$\implies 1 - a^2q^2 = \frac{1}{a^2z^2} \implies 1 - \frac{1}{a^2z^2} = a^2q^2$$

$$\implies \frac{a^2z^2 - 1}{a^2z^2} = a^2q^2$$

Solution of Problem 4 Continue...

$$\therefore \quad \mathrm{q}^2 = \frac{a^2z^2 - 1}{a^4z^2}$$

$$q = \frac{\sqrt{z^2a^2 - 1}}{a^2z}$$

Using value of q in $p = aq$

$$p = \frac{a\sqrt{z^2a^2 - 1}}{a^2z}$$

$$p = \frac{\sqrt{z^2a^2 - 1}}{az}$$

Consider, $p\ dx + q\ dy = dz$

$$\frac{\sqrt{z^2a^2 - 1}}{az}\ dx + \frac{\sqrt{z^2a^2 - 1}}{a^2z}\ dy = dz$$

$$adx + dy = \frac{a^2z}{\sqrt{z^2a^2 - 1}}dz$$

Type 2: PDE not involving independent variables

Solution of Problem 4 Continue...

Integrating we get,

$$ax + y = \int \frac{a^2 z}{\sqrt{z^2 a^2 - 1}} dz + b$$

For integration, we use substitution method

Put $a^2 z^2 - 1 = t$

$$\implies 2\mathrm{a}^2 z dz = dt \implies \mathrm{a}^2 z dz = \frac{dt}{2}$$

$$ax + y = \int \sqrt{t} \frac{dt}{2\sqrt{t}} + b$$

$$ax + y = \frac{2\sqrt{t}}{2}$$

$$ax + y = \sqrt{a^2 z^2 - 1} + b$$

This is Required Solution.

Special types of First Order Partial Differential Equations

Type 3: Separable Form

Type 3: Separable Form

The partial differential equation is said to be separable if it can be written in the form

$$\text{f(x , p)} = \text{g(y , q)} \tag{36}$$

$$\text{Let } F = f(x,p) - g(y,q) = 0$$
$$\frac{\partial F}{\partial x} = F_x = f_x$$
$$\frac{\partial F}{\partial y} = F_y = -g_x$$

Special types of First Order Partial Differential Equations

Type 3: Separable Form continue...

$$\frac{\partial F}{\partial z} = F_z = 0$$

$$\frac{\partial F}{\partial p} = F_p = f_p$$

$$\frac{\partial F}{\partial q} = F_q = -g_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{p\,f_p - q\,g_q} = \frac{-dp}{f_x} = \frac{-dq}{-g_y}$$

Special types of First Order Partial Differential Equations

Type 3: Separable Form Continue ...

Consider, $\dfrac{dx}{f_p} = \dfrac{-dp}{f_x}$

$\therefore \mathrm{f}_x dx + f_p dp = 0$

$\therefore \mathrm{d[f(x\ ,\ p)]} = 0$

Integrating we get,

$f(x, p) = a$

Similarly using this in (36)

$g(y, q) = a$

obtain expression for p and q from above two equations

using values of p and q in $p\ dx + q\ dy = dz$

Integrating we get required solution.

Type 3: Separable Form

Problem 1:

Find the complete integral of $p^2 + q^2 = x + y$

Solution

Given equation is

$$p^2 + q^2 = x + y$$

$$\therefore \mathrm{p}^2 - x = y - q^2$$

It is of the form

$$f(x, p) = g(y, q)$$

$$\text{Let } f(x, p) = p^2 - x = a$$

$$\therefore \mathrm{p}^2 = a + x$$

$$p = \sqrt{x + a}$$

Type 2: PDE not involving independent variables

Solution of Problem 1 Continue...

$$\text{Similarly, } g(y, q) = y - q^2 = a$$
$$\therefore \text{q}^2 = y - a$$
$$q = \sqrt{y - a}$$
$$\text{Consider, } p \ dx + q \ dy = dz$$
$$\sqrt{x + a} \ dx + \sqrt{y - a} \ dy = dz$$
$$(x + a)^{\frac{1}{2}} dx + (y - a)^{\frac{1}{2}} dy = dz$$

Integrating we get,

$$\frac{(x + a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(y - a)^{\frac{3}{2}}}{\frac{3}{2}} = z + b$$
$$(x + a)^{\frac{3}{2}} + (y - a)^{\frac{3}{2}} = \frac{3}{2}(z + b)$$

This is required solution.

Type 3: Separable Form

Problem 2:

Find the complete integral of $p^2 y(1+x^2) = qx^2$

Solution

Given equation is

$$p^2 y(1+x^2) = qx^2$$

$$\therefore \; \mathrm{p}^2 \left(\frac{1+x^2}{x^2}\right) = \frac{q}{y}$$

It is of the form

$$f(x,p) = g(y,q)$$

$$\text{Let } f(x,p) = p^2 \left(\frac{1+x^2}{x^2}\right) = a$$

Type 3: Separable Form

Solution of Problem 2 Continue...

$$\therefore \ p^2 = a\left(\frac{x^2}{1+x^2}\right)$$

$$p = x\sqrt{\frac{a}{1+x^2}}$$

Similarly, $g(y,q) = \dfrac{q}{y} = a$

$$q = ay$$

Consider, $p \ dx + q \ dy = dz$

$$\sqrt{\frac{a}{1+x^2}} \ xdx + ay \ dy = dz$$

$$\sqrt{a}\frac{xdx}{\sqrt{1+x^2}} + aydy = dz$$

Type 3: Separable Form

Solution of Problem 2 Continue...

Integrating we get,

$$\sqrt{a}\sqrt{1+x^2} + a\frac{y^2}{2} = z + b$$

$$2\sqrt{a}\sqrt{1+x^2} + ay^2 = 2z + b$$

This is required solution.

Special types of First Order Partial Differential Equations

Type 4: Claraut's Equation:

Type 4: Claraut's Equation:

The partial differential equation is of the form $z = px + qy + f(p, q)$ Where x and y are independent and z is dependent variable called as Claraut's Equation.

Now let $F = px + qy + f(x, p) - z = 0$

$$\frac{\partial F}{\partial x} = F_x = p$$

$$\frac{\partial F}{\partial y} = F_y = q$$

Special types of First Order Partial Differential Equations

Type 4: Claraut's Equation continue...

$$\frac{\partial F}{\partial z} = F_z = -1$$

$$\frac{\partial F}{\partial p} = F_p = x + f_p$$

$$\frac{\partial F}{\partial q} = F_q = y + f_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{dz}{xp + pf_p + yq + qf_q} = \frac{-dp}{p - p} = \frac{-dq}{q - q}$$

Special types of First Order Partial Differential Equations

Type 4: Claraut's Equation continue...

$$\text{Consider, Each Ratio} = \frac{dp}{0}$$

$$\therefore \text{dp} = 0$$

Integrating we get,

$$p = a$$

$$\text{Now Consider, Each Ratio} = \frac{dq}{0}$$

$$\therefore \text{dq} = 0$$

Integrating we get,

$$q = b$$

using values of p and q in $z = px + qy + f(p, q)$

$$z = ax + by + f(a, b)$$

Integrating we get required solution.

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Type 4: Claraut's Equation:

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Problem 1:

Find the complete integral of $(p+q)(z-px-qy)=1$

Solution

Given equation is

$$z - px - qy = \frac{1}{p+q}$$

$$z = px + qy + \frac{1}{p+q}$$

It is of the Claraut's Equation form

Hence its solution is,

$$z = ax + by + \frac{1}{a+b}$$

where a and b are constants.

Jacobi's Method

Jacobi's Auxiliary Equation

Show that the Jacobi's Auxiliary Equation is

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

Proof

This method is used for solving first order partial differential equation involving 3 or more independent variables.

Let

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \tag{37}$$

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

be first order partial differential equation where z is function of x_1, x_2 and x_3 and,

$$p_1 = \frac{\partial z}{\partial x_1}, \quad p_2 = \frac{\partial z}{\partial x_2}, \quad p_3 = \frac{\partial z}{\partial x_3}$$

such that

$$\mathrm{p}_1 \; dx_1 + p_2 \; dx_2 + p_3 \; dx_3 = dz \tag{38}$$

The Jacobi's method is same is same as that of Charpit's method. The main thing of Jacobi's method is to obtain two additional equations.

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \tag{39}$$

and

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2 \tag{40}$$

where a_1 and a_2 are arbitrary constants.
We find p_1, p_2, p_3 from (37) , (39) and (40) such that equation (38) will be integrable.

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

Differentiate (37) and (39) w.r.t. x_1

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial p_1}.\frac{\partial p_1}{\partial x_1} + \frac{\partial f}{\partial p_2}.\frac{\partial p_2}{\partial x_1} + \frac{\partial f}{\partial p_3}.\frac{\partial p_3}{\partial x_1} = 0 \qquad (41)$$

Similarly,

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial p_1}.\frac{\partial p_1}{\partial x_1} + \frac{\partial F_1}{\partial p_2}.\frac{\partial p_2}{\partial x_1} + \frac{\partial F_1}{\partial p_3}.\frac{\partial p_3}{\partial x_1} = 0 \qquad (42)$$

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

Multiply (41) by $\dfrac{\partial F_1}{\partial p_1}$ and (42) by $\dfrac{\partial f}{\partial p_1}$ then take (41) - (42)

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial x_1}\frac{\partial f}{\partial p_1}\right) + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial p_2}\frac{\partial f}{\partial p_1}\right)\frac{\partial p_2}{\partial x_1} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial p_3}\frac{\partial f}{\partial p_1}\right)\frac{\partial p_3}{\partial x_1} = 0 \tag{43}$$

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

Similarly, Differentiate (37) and (39) w.r.t. x_2
Replace $x_1 = x_2, p_1 = p_2, p_3 = constant$

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_1}{\partial x_2}\frac{\partial f}{\partial p_2}\right) + \left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_1}{\partial p_1}\frac{\partial f}{\partial p_2}\right)\frac{\partial p_1}{\partial x_2} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_1}{\partial p_3}\frac{\partial f}{\partial p_2}\right)\frac{\partial p_3}{\partial x_2} = 0 \tag{44}$$

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

Similarly, Differentiate (37) and (39) w.r.t. x_3
Replace $x_1 = x_3, p_1 = p_3, p_2 = constant$

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3} - \frac{\partial F_1}{\partial x_3}\frac{\partial f}{\partial p_3}\right) + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3} - \frac{\partial F_1}{\partial p_2}\frac{\partial f}{\partial p_3}\right)\frac{\partial p_2}{\partial x_3} + \left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3} - \frac{\partial F_1}{\partial p_1}\frac{\partial f}{\partial p_3}\right)\frac{\partial p_1}{\partial x_3} = 0 \tag{45}$$

Adding (43) , (44) and (45) and using

$$\frac{\partial p_2}{\partial x_1} = \frac{\partial}{\partial x_1}\frac{\partial z}{\partial x_2} = \frac{\partial^2 z}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2}\frac{\partial z}{\partial x_1} = \frac{\partial p_1}{\partial x_2}$$

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

using $\frac{\partial p_2}{\partial x_1} = \frac{\partial p_1}{\partial x_2}$, $\frac{\partial p_3}{\partial x_1} = \frac{\partial p_1}{\partial x_3}$ and $\frac{\partial p_3}{\partial x_2} = \frac{\partial p_2}{\partial x_3}$ we get,

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial x_1}\frac{\partial f}{\partial p_1}\right) + \left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_1}{\partial x_2}\frac{\partial f}{\partial p_2}\right) + \left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3} - \frac{\partial F_1}{\partial x_3}\frac{\partial f}{\partial p_3}\right) = 0$$

$$\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_1}{\partial p_r} - \frac{\partial F_1}{\partial x_r}\frac{\partial f}{\partial p_r}\right) = 0$$

Jacobi's Auxiliary Equation

Proof of Jacobi's Auxiliary Equation...

It's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

This relation is known as Jacobi's Auxiliary Equation. Similarly from (37) and (40) we get,

$$\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_2}{\partial x_r}\frac{\partial f}{\partial p_r}\right) = 0$$

After finding F-1$= a_1 and F_2 = a_2$.solving the equations

Solution of PDE by using Jacobi's Method

Problem 1:

Solve the PDE ${p_1}^3 + {p_2}^2 + p_3 = 1$ by Jacobi's method

Solution

Let

$$f = \mathrm{p}_1^3 + {p_2}^2 + p_3 - 1 = 0 \tag{46}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = 0$$
$$\frac{\partial f}{\partial x_2} = f_{x_2} = 0$$

Solution of PDE by using Jacobi's Method

Solution of Problem 1 Continue...

$$\frac{\partial f}{\partial x_3} = f_{x_3} = 0$$

$$\frac{\partial f}{\partial p_1} = f_{p_1} = 3{p_1}^2$$

$$\frac{\partial f}{\partial p_2} = f_{p_2} = 2p_2$$

$$\frac{\partial f}{\partial p_3} = f_{p_3} = -1$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

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Solution of Problem 1 Continue...

$$\frac{dx_1}{-3{p_1}^2} = \frac{dx_2}{2p_2} = \frac{dx_3}{-1} = \frac{dp_1}{0} = \frac{dp_2}{0} = \frac{dp_3}{0}$$

$$EachRatio = \frac{dp_1}{0} \implies \mathrm{dp}_1 = 0$$

Integrating,

$p_1 = a$.....where a is constant

$$\text{Similarly, } EachRatio = \frac{dp_2}{0} \implies \mathrm{dp}_2 = 0$$

Integrating,

$p_2 = b$.....where b is constant

Using $p_1 = a$ and $p_2 = b$ in equation (46)

$$a^3 + b^2 + p_3 = 1 \implies p_3 = 1 - a^3 - b^2$$

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Solution of Problem 1 Continue...

Using the values of p_1, p_2 and p_3 in

$$p_1\ dx_1 + p_2\ dx_2 + p_3\ dx_3 = dz$$

Consider, $p_1\ dx_1 + p_2\ dx_2 + p_3\ dx_3 = dz$

$$\therefore\ \ a\ \ \ \mathrm{dx}_1 + b\ dx_2 + (1 - a^3 - b^2)\ dx_3 = dz$$

Integrating we get,

$$a\ x_1 + b\ x_2 + (1 - a^3 - b^2)\ x_3 = z + c$$

Required Solution.

Solution of PDE by using Jacobi's Method

Problem 2:

Solve the PDE $2\,p_1\,x_1\,x_3 + 3\,p_2\,{x_3}^2 + {p_2}^2\,p_3 = 0$ by Jacobi's method

Solution

Let

$$f = 2\,\mathrm{p}_1\,x_1\,x_3 + 3\,p_2\,{x_3}^2 + {p_2}^2\,p_3 = 0 \tag{47}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = 2\,p_1\,x_3$$

$$\frac{\partial f}{\partial x_2} = f_{x_2} = 0$$

Solution of PDE by using Jacobi's Method

Solution of Problem 2 Continue...

$$\frac{\partial f}{\partial x_3} = f_{x_3} = 2\,p_1\,x_1 + 6\,p_2\,x_3$$

$$\frac{\partial f}{\partial p_1} = f_{p_1} = 2\,x_1\,x_3$$

$$\frac{\partial f}{\partial p_2} = f_{p_2} = 3\,{x_3}^2 + 2\,p_2\,p_3$$

$$\frac{\partial f}{\partial p_3} = f_{p_3} = {p_2}^2$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

Solution of Problem 2 Continue...

$$\frac{dx_1}{-(2\ x_1\ x_3)} = \frac{dx_2}{-(3\ {x_3}^2 + 2\ p_2\ p_3)} = \frac{dx_3}{-{p_2}^2} = \frac{dp_1}{2\ p_1\ x_3} = \frac{dp_2}{0} = \frac{dp_3}{2\ p_1\ x_1 + 6\ p_2\ x_3}$$

$$EachRatio = \frac{dp_2}{0} \implies \mathrm{dp}_2 = 0$$

Integrating,

$p_2 = a$.....where a is constant

Similarly, Consider,

$$\frac{dx_1}{-(2\, x_1\, x_3)} = \frac{dp_1}{2\, p_1\, x_3}$$

$$\frac{dx_1}{-x_1} = \frac{dp_1}{p_1}$$

Integrating,

$$-\ln x_1 = \ln p_1 - \ln b$$

Solution of Problem 2 Continue...

$\ln x_1 + \ln p_1 = \ln b$

$\ln x_1 p_1 = \ln b \implies x_1 p_1 = b$

$p_1 = \dfrac{b}{x_1}$.....where b is constant

Using $p_1 = \dfrac{b}{x_1}$ and $p_2 = a$ in equation (47)

$2\,\dfrac{b}{x_1}\,x_1\,x_3 + 3\,a\,{x_3}^2 + a^2\,p_3 = 0$

$a^2\,p_3 = -2\,b\,x_3 - 3\,a\,{x_3}^2$

$p_3 = -\dfrac{1}{a^2}(2\,b\,x_3 + 3\,a\,{x_3}^2)$

Solution of PDE by using Jacobi's Method

Solution of Problem 2 Continue...

Using the values of p_1, p_2 and p_3 in

$$p_1\ dx_1 + p_2\ dx_2 + p_3\ dx_3 = dz$$

Consider,

$$p_1\ dx_1 + p_2\ dx_2 + p_3\ dx_3 = dz$$

$$\therefore\quad \frac{b}{x_1}\ dx_1 + a\ dx_2 - \frac{1}{a^2}(2\,b\,x_3 + 3\,a\,{x_3}^2)dx_3 = dz$$

Integrating we get,

$$b\ln x_1 + a\,x_2 - \frac{1}{a^2}\left(2\,b\frac{{x_3}^2}{2} + 3\,a\frac{{x_3}^3}{3}\right) = z + c$$

$$b\ln x_1 + a\,x_2 - \frac{b}{a^2}{x_3}^2 - \frac{{x_3}^3}{a} = z + c$$

Required Solution.

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Problem 3:

Solve the PDE $p_1\, x_1 + p_2\, x_2 = {p_3}^2$ by Jacobi's method

Solution

Let

$$f = p_1\, x_1 + p_2\, x_2 - {p_3}^2 = 0 \tag{48}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = p_1$$
$$\frac{\partial f}{\partial x_2} = f_{x_2} = p_2$$

Solution of Problem 3 Continue...

$$\frac{\partial f}{\partial x_3} = f_{x_3} = 0$$

$$\frac{\partial f}{\partial p_1} = f_{p_1} = x_1$$

$$\frac{\partial f}{\partial p_2} = f_{p_2} = x_2$$

$$\frac{\partial f}{\partial p_3} = f_{p_3} = -2p_3$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

Solution of PDE by using Jacobi's Method

Solution of Problem 3 Continue...

$$\frac{dx_1}{-x_1} = \frac{dx_2}{-x_2} = \frac{dx_3}{-(-2p_3)} = \frac{dp_1}{p_1} = \frac{dp_2}{p_2} = \frac{dp_3}{0}$$

$$EachRatio = \frac{dp_3}{0} \implies \mathrm{dp}_3 = 0$$

Integrating,

$p_3 = a$.....where a is constant

Similarly, Consider,

$$\frac{dx_1}{-x_1} = \frac{dp_1}{p_1}$$

Integrating,

$$-\ln x_1 = \ln p_1 - \ln b$$

Solution of PDE by using Jacobi's Method

Solution of Problem 3 Continue...

$$\ln x_1 + \ln p_1 = \ln b$$

$$\ln x_1 p_1 = \ln b \implies x_1 p_1 = b$$

$$p_1 = \frac{b}{x_1}$$ .....where b is constant

Using $p_1 = \dfrac{b}{x_1}$ and $p_3 = a$ in equation (48)

$$2\,\frac{b}{x_1}\,x_1 + p_2\,x_2 - a^2 = 0$$

$$p_2\,x_2 = a^2 - b$$

$$p_2 = \frac{a^2 - b}{x_2}$$

Solution of PDE by using Jacobi's Method

Solution of Problem 3 Continue...

Using the values of p_1, p_2 and p_3 in

$p_1 \ dx_1 + p_2 \ dx_2 + p_3 \ dx_3 = dz$

Consider,

$$p_1 \ dx_1 + p_2 \ dx_2 + p_3 \ dx_3 = dz$$

$$\therefore \quad \frac{b}{x_1} \ dx_1 + \frac{a^2 - b}{x_2} \ dx_2 + a dx_3 = dz$$

Integrating we get,

$b \ln x_1 + (a^2 - b) \ln x_2 + a \, x_3 = z + c$

$$b \ln x_1 + (a^2 - b) \ln x_2 + a \, x_3 = z + c$$

Required Solution.

Solution of PDE by using Jacobi's Method

Problem 4:

Solve the PDE $p_1\, p_2\, p_3 = z^3\, x_1\, x_2\, x_3$ by Jacobi's method

Solution

Let

$$p_1\, p_2\, p_3 = z^3\, x_1\, x_2\, x_3$$

Dividing by z^3

$$\left(\frac{1}{z}\, p_1\right)\left(\frac{1}{z}\, p_2\right)\left(\frac{1}{z}\, p_3\right) = x_1\, x_2\, x_3$$

Put $u = \log z$

Differentiate w.r.t. x_1, we get

$$\therefore \quad \frac{\partial u}{\partial x_1} = \frac{1}{z}\frac{\partial z}{\partial x_1} = \frac{1}{z}\, p_1$$

Solution of PDE by using Jacobi's Method

Solution of Problem 4 Continue...

Take $P_1 = \frac{1}{z}\, p_1, \quad P_2 = \frac{1}{z}\, p_2, \quad P_3 = \frac{1}{z}\, p_3$

Equation becomes,

$$P_1\, P_2\, P_3 = x_1\, x_2\, x_3$$

$$\mathrm{f} = \mathrm{P}_1\, P_2\, P_3 - x_1\, x_2\, x_3 = 0 \tag{49}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = -x_2\, x_3,$$
$$\frac{\partial f}{\partial x_2} = f_{x_2} = -x_1\, x_3,$$
$$\frac{\partial f}{\partial x_3} = f_{x_3} = -x_1\, x_2;$$

Solution of PDE by using Jacobi's Method

Solution of Problem 4 Continue...

$$\frac{\partial f}{\partial P_1} = f_{P_1} = P_2 P_3$$

$$\frac{\partial f}{\partial P_2} = f_{P_2} = P_1 P_3$$

$$\frac{\partial f}{\partial P_3} = f_{P_3} = P_1 P_2$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{P_1}} = \frac{dx_2}{-f_{P_2}} = \frac{dx_3}{-f_{P_3}} = \frac{dP_1}{f_{x_1}} = \frac{dP_2}{f_{x_2}} = \frac{dP_3}{f_{x_3}}$$

$$\frac{dx_1}{-P_2 P_3} = \frac{dx_2}{-P_1 P_3} = \frac{dx_3}{-P_1 P_2} = \frac{dP_1}{-x_2 x_3} = \frac{dP_2}{-x_1 x_3} = \frac{dP_3}{-x_1 x_2}$$

Solution of Problem 4 Continue...

Consider

$$\frac{dx_1}{-P_2P_3} = \frac{dP_1}{-x_2x_3}$$

$$\therefore \quad \frac{P_1\,dx_1}{P_1P_2P_3} = \frac{dP_1}{x_2x_3}$$

$$\therefore \quad \frac{P_1\,dx_1}{x_1x_2x_3} = \frac{dP_1}{x_2x_3} \quad \text{..... By Equation (49)}$$

$$\therefore \quad \frac{dx_1}{x_1} = \frac{dP_1}{P_1}$$

Integrating,

$$\therefore \quad \log x_1 = \log P_1 - \log a$$

$$\therefore \quad \log x_1 + \log a = \log P_1$$

$$P_1 = ax_1 \text{.....where a is constant}$$

Solution of PDE by using Jacobi's Method

Solution of Problem 4 Continue...

Similarly, Consider,

$$\frac{dx_2}{-P_1P_3} = \frac{dP_2}{-x_1x_3}$$

$$\therefore \quad \frac{P_2dx_2}{P_1P_2P_3} = \frac{dP_2}{x_1x_3}$$

$$\therefore \quad \frac{P_2dx_2}{x_1x_2x_3} = \frac{dP_2}{x_1x_3} \text{... by equation (49)}$$

$$\therefore \quad \frac{dx_2}{x_2} = \frac{dP_2}{P_2}$$

Integrating,

$$\therefore \quad \log x_2 = \log P_2 - \log b$$

$$\therefore \quad \log x_2 + \log b = \log P_2$$

$$P_2 = bx_2 \text{.....where b is constant}$$

Solution of PDE by using Jacobi's Method

Solution of Problem 4 Continue...

Using $P_1 = ax_1$ and $P_2 = bx_2$ in equation (49)

$$\therefore \text{a} \quad \text{x}_1 \, b \, x_2 \, P_3 - x_1 x_2 x_3 = 0$$

$$\therefore \text{a} \quad \text{x}_1 \, b \, x_2 \, P_3 = x_1 x_2 x_3$$

$$\therefore \text{P}_3 = \frac{x_3}{ab}$$

$$P_3 = \frac{x_3}{ab}$$

Using the values of p_1, p_2 and p_3 in in

$$P_1 \;\; dx_1 + P_2 \;\; dx_2 + P_3 \;\; dx_3 = dz$$

$$\therefore \quad \text{a x}_1 \;\; dx_1 + bx_2 \;\; dx_2 + \frac{x_3}{ab} dx_3 = dz$$

Integrating we get,

$$\frac{a{x_1}^2}{2} + \frac{b{x_2}^2}{2} + \frac{{x_3}^2}{2ab} z + c$$

$a^2 b{x_1}^2 + ab^2{x_2}^2 + {x_3}^2 = 2abz + c$... Required Solution.

Solution of PDE by using Jacobi's Method

Problem 5:

Solve the PDE $p^2 x + q^2 y = z$ by Jacobi's method

Solution

Jacobi's method is used for solving first order partial differential equation involving 3 or more independent variables. Here x and y are independent and z is dependent variable. So we consider z as independent variable

if and only if $u(x, y, z) = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \implies \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p = 0$$

Solution of PDE by using Jacobi's Method

Solution of Problem 5 Continue...

Let

$$u_1 = \frac{\partial u}{\partial x}, \quad u_2 = \frac{\partial u}{\partial y}, \quad u_3 = \frac{\partial u}{\partial z}$$

$$\therefore \ u_1 + u_3 \ p = 0$$

$$\therefore \ p = - \ \frac{u_1}{u_3}$$

$$\text{Similarly, } q = -\frac{u_2}{u_3}$$

Using this in (1)

$$\frac{{u_1}^2}{{u_3}^2} x + \frac{{u_2}^2}{{u_3}^2} y = z$$

$$\therefore \ u_1^2 x + {u_2}^2 y = {u_3}^2 z$$

Solution of Problem 5 Continue...

Let

$$f = u_1^2\, x + {u_2}^2\, y - {u_3}^2\, z = 0 \tag{50}$$

$$\frac{\partial f}{\partial x} = f_x = {u_1}^2,$$
$$\frac{\partial f}{\partial y} = f_y = {u_2}^2,$$
$$\frac{\partial f}{\partial z} = f_z = -{u_3}^2;$$
$$\frac{\partial f}{\partial u_1} = f_{u_1} = 2u_1\, x$$
$$\frac{\partial f}{\partial u_2} = f_{u_2} = 2u_2\, y \text{ and } \frac{\partial f}{\partial u_3} = f_{u_3} = -2u_3\, z$$

Solution of Problem 5 Continue...

Jacobi's Auxiliary equation is,

$$\frac{dx}{-f_{u_1}} = \frac{dy}{-f_{u_2}} = \frac{dz}{-f_{u_3}} = \frac{du_1}{f_x} = \frac{du_2}{f_y} = \frac{du_3}{f_z}$$

$$\frac{dx}{-2u_1\,x} = \frac{dy}{-2u_2\,y} = \frac{dz}{-2u_3\,z} = \frac{du_1}{{u_1}^2} = \frac{du_2}{{u_2}^2} = \frac{du_3}{-{u_3}^2}$$

Consider

$$\frac{dx}{-2u_1\,x} = \frac{du_1}{{u_1}^2}$$
$$\frac{dx}{-2\,x} = \frac{du_1}{u_1}$$

Solution of Problem 5 Continue...

Integrating,

$$\therefore \quad -\frac{1}{2}\log x = \log u_1 - \log a$$

$$\therefore \quad \log {u_1}^2 + \log x = \log a$$

$$\therefore \quad \log {u_1}^2 x = \log a$$

$$\therefore \quad {u_1}^2 x = a$$

$$u_1 = \sqrt{\frac{a}{x}} \text{.....where a is constant}$$

Similarly, Consider,

$$\frac{dy}{-2u_2\, y} = \frac{du_2}{{u_2}^2}$$

$$\frac{dy}{-2\, y} = \frac{du_2}{u_2}$$

Solution of Problem 5 Continue...

Integrating,

$$\therefore \quad -\frac{1}{2}\log y = \log u_2 - \log b$$

$$\therefore \quad \log {u_2}^2 + \log y = \log b$$

$$\therefore \quad \log {u_2}^2 y = \log b \therefore \quad {u_2}^2 y = b$$

$$u_2 = \sqrt{\frac{b}{y}} \text{.....where b is constant}$$

Using $u_1 = \sqrt{\frac{a}{x}}$ and $u_2 = \sqrt{\frac{b}{y}}$ in equation (50)

$$\therefore \quad \frac{a}{x}x + \frac{b}{y}y - {u_3}^2 z = 0 \implies \therefore \text{u}_3^2 z = a + b$$

$$u_3 = \sqrt{\frac{a+b}{z}}$$

Solution of PDE by using Jacobi's Method

Solution of Problem 5 Continue...

Using the values of u_1, u_2 and u_3 in in

$$u_1\ dx + u_2\ dy + u_3\ dz = du$$

$$\therefore\ \sqrt{\frac{a}{x}}\ dx + \sqrt{\frac{b}{y}}\ dy + \sqrt{\frac{a+b}{z}}\ dz = du$$

$$\therefore\ \sqrt{a}\ \frac{dx}{\sqrt{x}} + \sqrt{b}\ \frac{dy}{\sqrt{y}} + \sqrt{a+b}\ \frac{dz}{\sqrt{z}} = du$$

Integrating we get,

$$2\sqrt{ax} + 2\sqrt{by} + 2\sqrt{(a+b)z} = u + c$$

But $u(x, y, z) = 0$

$$\sqrt{ax} + \sqrt{by} + \sqrt{(a+b)z} = c$$

Required Solution.